Lesson 17: Relative Extrema and Critical Numbers

Def. relative extrema: minimums or maximums anywhere on a function (can have more than one or none of each)

* Function must be defined at the point. (Not a hole or a vertical asymptote.)

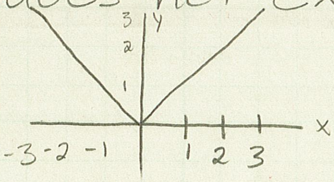
Def. critical number: any x-value where the derivative of a function is 0, undefined, or does not exist

* Must be in domain of the function

* Relative extrema only occur at critical numbers, so to find relative extrema, we first must find the critical numbers.
However, just because an x-value is a critical number, it does not automatically mean there is a min or max there.

Note: If a function is too "steep," the derivative does not exist at a point.

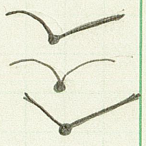
The classic example is $f(x) = |x|$ because $f'(x)$ does not exist at $x=0$.



The derivative is the slope of the tangent line, so for
- $x < 0$, $f'(x) = -1$
- $x > 0$, $f'(x) = 1$.

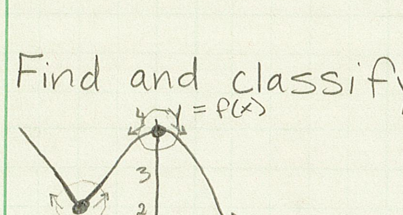
The derivative is defined by the limit, so at $x=0$, $-1 \neq 1$ means the limit does not exist.

We call points like this
- kinks
- cusps
- or corners



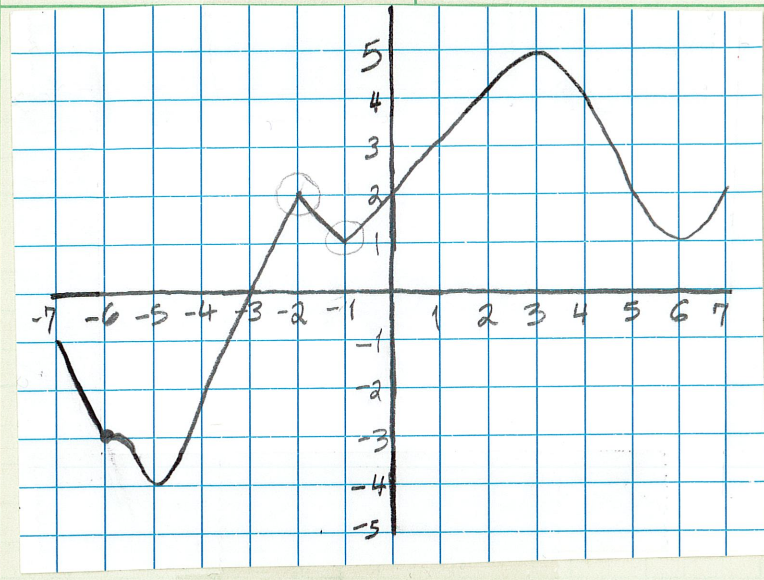
* You only have to look for these on graphs.* (not in functions)

Ex. 1 Find and classify the critical numbers as min, max, or neither.



- At $(-2, 2)$, we have a rel. min because the derivative DNE, and all $f(x)$ values near $x=-2$ are greater than 2.
- At $(0, 4)$, we have a rel. max because the derivative $= 0$, and all $f(x)$ values near $x=0$ are less than 4.
- $(2,1)$ is not a min or max because to the left, the y-values are bigger, but to the right, the y-values are smaller.

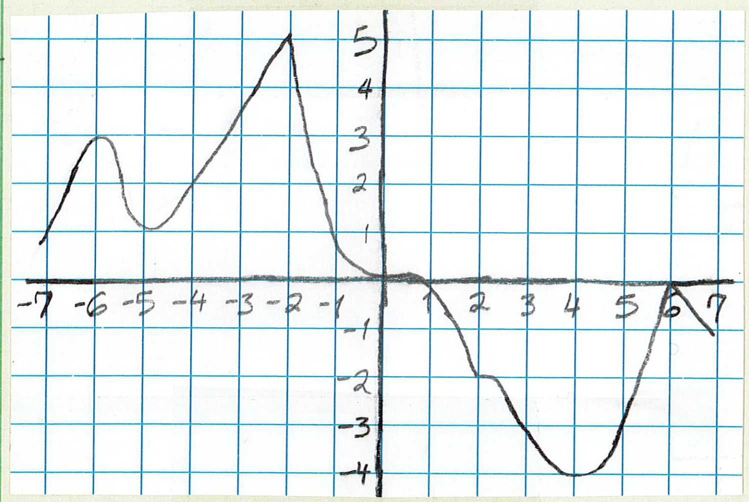
Ex. 2



Find and classify the critical numbers as min, max, or neither.

Critical Number	Derivative DNE or $=0$	Rel min/max? (and point)
$x=-6$	DNE	Neither
$x=-5$	$=0$	Rel Min $(-5,-4)$
$x=-2$	DNE	Rel Max $(-2,2)$
$x=-1$	DNE	Rel Min $(-1,1)$
$x=3$	$=0$	Rel Max $(3,5)$
$x=6$	$=0$	Rel Min $(6,1)$

Ex. 3



Critical Number	Derivative DNE or $=0$?	Rel min/max? (and point)
$x=-6$	$=0$	Rel Max $(-6,3)$
$x=-5$	$=0$	Rel Min $(-5,1)$
$x=-2$	DNE	Rel Max $(-2,5)$
$x=0$	$=0$	Neither
$x=2$	DNE	Neither
$x=4$	$=0$	Rel Min $(4,-4)$
$x=6$	DNE	Rel Max $(6,0)$

Find the critical numbers for the following functions:

Ex. 4 $y = x^2 - 3x$

⓪ Domain: $(-\infty, \infty)$

① Differentiate: $y' = 2x - 3$

② Find when $y' = 0$ or is undefined: $y' = 2x-3$ defined everywhere

$$0 = 2x - 3$$
$$3 = 2x$$
$$\frac{3}{2} = x$$

③ Check that x-values are in the domain: $x = \frac{3}{2}$ in domain ✓

$\boxed{x = \frac{3}{2}}$ is the only critical number. In LON-CAPA:

$x_1 = \boxed{3/2}$

$x_2 = \boxed{\text{NONE}}$

Ex.5 $g(x) = \frac{x^4-3}{x^2}$

⓪ $g(x)$ is undefined when denominator $=0$, so when $x^2=0$
$\Rightarrow x=0$. This means domain: $(-\infty,0)\cup(0,\infty)$
or rather $x \neq 0$

① Differentiate: First, rewrite $g(x) = \frac{x^4}{x^2} - \frac{3}{x^2}$
$= x^2 - 3x^{-2}$

Derivative: $g'(x) = 2x - 3(-2x^{-2-1})$
$= 2x + 6x^{-3}$
$= 2x + \frac{6}{x^3}$

② $g'(x)$ is undefined when $x=0$ because of the x^3 in the denominator, but $x=0$ is not in the domain, so it's not a critical number.
Set $=0$: $g'(x) = 2x + \frac{6}{x^3} = 0$

$$2x = -\frac{6}{x^3}$$
$$2x^4 = -6$$
$$x^4 = -3 \quad \text{never happens!}$$

③ $\boxed{\text{No critical numbers}}$ In LON-CAPA: $x_1 = \boxed{\text{NONE}}$
$x_2 = \boxed{\text{NONE}}$

Ex.6 Find the critical numbers for $y = 6\cos(5x) + 15x$ on the interval of $(0,\pi)$.

⓪ Domain: $(-\infty,\infty)$

① Differentiate: $y' = 6\underbrace{\frac{d}{dx}(\cos(5x))}$ $+15$

Chain: $Out = \cos(x)$ $In = 5x$
$Out' = -\sin(x)$ $In' = 5$

$$y' = 6(-\sin(5x)\cdot 5) + 15$$
$$= -30\sin(5x) + 15$$

② y' is defined everywhere, so set $y'=0$ and solve:

$$0 = -30\sin(5x) + 15$$
$$30\sin(5x) = 15$$
$$\sin(5x) = \frac{1}{2}$$
$$5x = \sin^{-1}\left(\frac{1}{2}\right)$$
$= \frac{\pi}{6}$ or $= \frac{5\pi}{6}$

$5x = \frac{\pi}{6} + 2\pi n$
$x = \frac{\pi}{30} + \frac{2}{5}\pi n$
$\frac{2}{5} = \frac{12}{30}$

$5x = \frac{5\pi}{6} + 2\pi n$
$x = \frac{5\pi}{30} + \frac{2}{5}\pi n$

③ Only want critical numbers in $(0,\pi)$: $\boxed{\frac{\pi}{30}, \frac{13\pi}{30}, \frac{5\pi}{6}, \frac{\pi}{6}, \frac{17\pi}{30}, \frac{29\pi}{30}}$

$x = \frac{\pi}{30}$
$x = \frac{\pi}{30} + \frac{12\pi}{30} = \frac{13\pi}{30}$
$x = \frac{13\pi}{30} + \frac{12\pi}{30} = \frac{25\pi}{30} = \frac{5\pi}{6}$
$x = \frac{25\pi}{30} + \frac{12\pi}{30} = \cancel{\frac{37\pi}{30}} > \pi$

$x = \frac{5\pi}{30} = \frac{\pi}{6}$
$x = \frac{5\pi}{30} + \frac{12\pi}{30} = \frac{17\pi}{30}$
$x = \frac{17\pi}{30} + \frac{12\pi}{30} = \frac{29\pi}{30}$
$x = \frac{29\pi}{30} + \frac{12\pi}{30} = \cancel{\frac{41\pi}{30}} > \pi$

Ex.7 $y = 2x^3 + x^2 - 7x + 2$

⓪ Domain: $(-\infty, \infty)$

① Differentiate: $y' = 6x^2 + 2x - 7$

② y' defined everywhere, so solve $y' = 0$:

$0 = 6x^2 + 2x - 7 \rightarrow$ Need quadratic formula

$0 = ax^2 + bx + c$

$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$x = \frac{-2 \pm \sqrt{(2)^2 - 4(6)(-7)}}{2(6)}$$

$$x = \frac{-2 \pm \sqrt{4 + 168}}{12}$$

$$x = \frac{-2 \pm \sqrt{172}}{12} \quad \left(= \frac{-2 \pm \sqrt{4 \cdot 43}}{12} = \frac{-2 \pm \sqrt{4}\sqrt{43}}{12} = \frac{-2 \pm 2\sqrt{43}}{12} = \frac{2(-1 \pm \sqrt{43})}{12} = \frac{-1 \pm \sqrt{43}}{6} \right)$$

③ Critical Numbers: $\boxed{x_1 = \dfrac{-1 - \sqrt{43}}{6} \qquad x_2 = \dfrac{-1 + \sqrt{43}}{6}}$

Ex.8 $y = 2x^4 e^{3x-1}$

⓪ Domain: $(-\infty, \infty)$

① Differentiate: $y' = \dfrac{d}{dx}[2x^4] \cdot e^{3x-1} + 2x^4 \cdot \underbrace{\dfrac{d}{dx}[e^{3x-1}]}_{\text{chain}}$

$$= 8x^3 \cdot e^{3x-1} + 2x^4(3e^{3x-1})$$

$$= 8x^3 e^{3x-1} + 6x^4 e^{3x-1}$$

② y' is defined everywhere, so solve $y' = 0$.

Factor out gcf, do not subtract and divide

$$0 = 8x^3 e^{3x-1} + 6x^4 e^{3x-1}$$

$$0 = \underbrace{2x^3}_{a}\underbrace{e^{3x-1}}_{b}\underbrace{(4 + 3x)}_{c}$$

a, b, or c must be 0 for the product to be 0.

$0 = 2x^3$	$0 = e^{3x-1}$	$0 = 4 + 3x$
$0 = x$	Never happens	$-4 = 3x$
		$-\frac{4}{3} = x$

③ $\boxed{x_1 = 0 \text{ and } x_2 = -\frac{4}{3}}$ are in the domain, so both are critical numbers.