Lesson 17: Relative Extrema and Critical Numbers

**Def. relative extrema:** minimums or maximums anywhere on a function (can have more than one or none of each)

* Function must be defined at the point. (Not a hole or a vertical asymptote.)

**Def. critical number:** any x-value where the derivative of a function is 0, undefined, or does not exist.

* Must be in domain of the function

* Relative extrema only occur at critical numbers, so to find relative extrema, we first must find the critical numbers. However, just because an x-value is a critical number, it does not automatically mean there is a min or max there.

**Note:** If a function is too "steep," the derivative does not exist at a point.

The classic example is $f(x) = |x|$ because $f'(x)$ does not exist at $x = 0$.

The derivative is the slope of the tangent line, so for:
- $x < 0$, $f'(x) = -1$
- $x > 0$, $f'(x) = 1$.

The derivative is defined by the limit so at $x = 0$, $-1 \neq 1$ means the limit does not exist.

We call points like this "kinks" or "cusps" or "corners".

* You only have to look for these on graphs.*

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Ex. 1 Find and classify the critical numbers as min, max, or neither.

- **At** $(-2, 2)$, we have a rel. min because the derivative DNE and all $f(x)$ values near $x = -2$ are greater than 2.
- **At** $(0, 4)$, we have a rel. max because the derivative = 0 and all $f(x)$ values near $x = 0$ are less than 4.
- $(2, 1)$ is not a min or max because to the left, the y-values are bigger and to the right, the y-values are smaller.
**Ex. 2**

Find and classify the critical numbers as min, max, or neither.

<table>
<thead>
<tr>
<th>Critical Number</th>
<th>Derivative DNE or = 0</th>
<th>Rel min/max? (and point)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x = -6 )</td>
<td>DNE</td>
<td>Neither</td>
</tr>
<tr>
<td>( x = -5 )</td>
<td>= 0</td>
<td>Rel Min ((-5, -4))</td>
</tr>
<tr>
<td>( x = -3 )</td>
<td>DNE</td>
<td>Rel Max ((-2, 0))</td>
</tr>
<tr>
<td>( x = -1 )</td>
<td>DNE</td>
<td>Rel Min ((-1, 1))</td>
</tr>
<tr>
<td>( x = 3 )</td>
<td>= 0</td>
<td>Rel Max ((3, 5))</td>
</tr>
<tr>
<td>( x = 6 )</td>
<td>= 0</td>
<td>Rel Min ((6, 0))</td>
</tr>
</tbody>
</table>

**Ex. 3**

<table>
<thead>
<tr>
<th>Critical Number</th>
<th>Derivative DNE or = 0?</th>
<th>Rel min/max? (and point)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x = -6 )</td>
<td>= 0</td>
<td>Rel Max ((-6, 3))</td>
</tr>
<tr>
<td>( x = -5 )</td>
<td>= 0</td>
<td>Rel Min ((-5, 1))</td>
</tr>
<tr>
<td>( x = -2 )</td>
<td>DNE</td>
<td>Rel Max ((-2, 5))</td>
</tr>
<tr>
<td>( x = 0 )</td>
<td>= 0</td>
<td>Neither</td>
</tr>
<tr>
<td>( x = 2 )</td>
<td>DNE</td>
<td>Neither</td>
</tr>
<tr>
<td>( x = 4 )</td>
<td>= 0</td>
<td>Rel Min ((4, 4))</td>
</tr>
<tr>
<td>( x = 6 )</td>
<td>DNE</td>
<td>Rel Max ((6, 0))</td>
</tr>
</tbody>
</table>

Find the critical numbers for the following functions:

**Ex. 4**

\[ y = x^2 - 3x \]

1. **Domain**: \((-\infty, \infty)\)
2. **Differentiate**: \[ y' = 2x - 3 \]
3. **Find when \( y' = 0 \) or is undefined**: \[ y' = 2x - 3 \] defined everywhere

   \[ 0 = 2x - 3 \]
   \[ 2 = 2x \]
   \[ \frac{1}{2} = x \]

   \( x = \frac{3}{2} \) in domain

**Check that x-values are in the domain**: \( x = \frac{3}{2} \) is the only critical number. In LON-CAPA:

\[ x_1 = \frac{3}{2} \]
\[ x_2 = \text{NONE} \]
Ex. 5
\[ g(x) = \frac{x^4 - 3}{x^2} \]

\( g(x) \) is undefined when denominator = 0, so when \( x^2 = 0 \)
\[ \Rightarrow x = 0. \] This means domain: \(( -\infty, 0) \cup (0, \infty)\)
or rather \( x \neq 0 \)

\( 0 \) Differentiate: First, rewrite \( g(x) = \frac{x^4}{x^2} - \frac{3}{x^2} \)
\[ = x^2 - 3x^{-2} \]

Derivative: \( g'(x) = 2x - 3(-2x^{-3}) \)
\[ = 2x + \frac{6}{x^3} \]

\( 2 \) \( g'(x) \) is undefined when \( x = 0 \) because of the \( x^2 \) in the denominator, but \( x = 0 \) is not in the domain, so it's not a critical number.

Set = 0: \( g'(x) = 2x + \frac{6}{x^3} = 0 \)
\[ 2x = -\frac{6}{x^3} \]
\[ 2x^4 = -6 \]
\[ x^4 = -3 \text{ never happens!} \]

\( 3 \) No critical numbers

In LON-CAPA: \( x_1 = \text{NONE} \)
\[ x_2 = \text{NONE} \]

Ex. 6
Find the critical numbers for \( y = 6 \cos(5x) + 15x \) on the interval of \(( 0, \pi)\).

\( 0 \) Domain: \(( -\infty, \infty)\)

\( 0 \) Differentiate: \( y' = 6 \frac{d}{dx}(\cos(5x)) + 15 \)
Chain: \( \text{Out} = \cos(x) \quad \text{In} = 5x \)
\[ \text{Out'} = -\sin(x) \quad \text{In'} = 5 \]

\[ y' = 6(-\sin(5x) \cdot 5) + 15 \]
\[ = -30\sin(5x) + 15 \]

\( 2 \) \( y' \) is defined everywhere, so set \( y' = 0 \) and solve:

\[ 0 = -30\sin(5x) + 15 \]
\[ 30\sin(5x) = 15 \]
\[ \sin(5x) = \frac{1}{2} \]
\[ 5x = \sin^{-1}(\frac{1}{2}) \]
\[ = \frac{\pi}{6} \quad \text{or} = \frac{5\pi}{6} \]

\[ x = \frac{\pi}{30} + \frac{2\pi}{3} \quad x = \frac{5\pi}{30} + \frac{2\pi}{3} \]

\( 3 \) Only want critical numbers in \(( 0, \pi)\):
\[ x = \frac{\pi}{30}, \frac{12\pi}{30} = \frac{13\pi}{30} \]
\[ x = \frac{17\pi}{30}, \frac{12\pi}{30} = \frac{29\pi}{30} \]
\[ x = \frac{23\pi}{30}, \frac{12\pi}{30} = \frac{35\pi}{30} \rightarrow \pi \]

\[ x = \frac{2\pi}{30}, \frac{12\pi}{30} = \frac{14\pi}{30} \rightarrow \pi \]

\[ x = \frac{5\pi}{30}, \frac{12\pi}{30} = \frac{17\pi}{30} \]
\[ x = \frac{11\pi}{30}, \frac{12\pi}{30} = \frac{29\pi}{30} \]
\[ x = \frac{23\pi}{30}, \frac{12\pi}{30} = \frac{35\pi}{30} \rightarrow \pi \]
Ex. 7 \[ y = 2x^3 + x^2 - 7x + 2 \]

\(\text{1. Domain: } (-\infty, \infty)\)

\(\text{2. Differentiate: } y' = 6x^2 + 2x - 7\)

\(\text{3. } y' \text{ defined everywhere, so solve } y' = 0:\)

\[ 0 = 6x^2 + 2x - 7 \quad \text{Need quadratic formula} \]

\[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

\[ x = \frac{-2 \pm \sqrt{4 + 4 \cdot 168}}{12} \]

\[ x = \frac{-2 \pm 14 \cdot 43}{12} \]

\[ x_1 = -\frac{1 - 43}{6}, \quad x_2 = -\frac{1 + 43}{6} \]

\(\text{Critical Numbers: } x_1 = -\frac{1 - 43}{6}, \quad x_2 = -\frac{1 + 43}{6}\)

Ex. 8 \[ y = 2x^4 e^{3x-1} \]

\(\text{1. Domain: } (-\infty, \infty)\)

\(\text{2. Differentiate: } y' = \frac{d}{dx} [2x^4] e^{3x-1} + 2x^4 \cdot \frac{d}{dx} [e^{3x-1}] \]

\[ = 8x^3 e^{3x-1} + 6x^4 e^{3x-1} \]

\(\text{2. } y' \text{ is defined everywhere, so solve } y' = 0.\)

\[ 0 = 8x^3 e^{3x-1} + 6x^4 e^{3x-1} \]

\[ \text{Factor out } 0 = 2x^3 e^{3x-1} (4 + 3x) \]

\[ a, b, \text{ or } c \text{ must be 0 for the product to be 0.} \]

\[ 0 = 2x^3 \quad 0 = e^{3x-1} \]

\[ 0 = x, \quad \text{Never happens} \]

\[ -4 = 3x \]

\[ -\frac{4}{3} = x \]

\(\text{3. } x_1 = 0 \text{ and } x_2 = -\frac{4}{3} \text{ are in the domain, so both are critical numbers.}\)