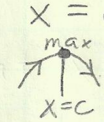
Lesson 18: Increasing / Decreasing Functions; First Derivative Test

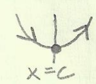
We are going to use the first derivative ($f'(x)$) of a function to figure out information about the function itself ($f(x)$).

Def. $f(x)$ is <u>increasing</u> when $f'(x) > 0$.
$f(x)$ is <u>decreasing</u> when $f'(x) < 0$.

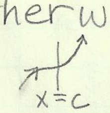
* <u>First Derivative Test</u> (1^{st} DT).

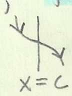
Let $x = c$ be a critical number for $f(x)$.

- If $f'(x)$ goes from positive to negative at $x = c$, (i.e. $f(x)$ goes from increasing to decreasing 

) $f(x)$ has a relative maximum at $x = c$.
- If $f'(x)$ goes from negative to positive at $x = c$, (i.e. $f(x)$ goes from decreasing to increasing 

) $f(x)$ has a relative minimum at $x = c$.
- Otherwise, $f(x)$ has neither min nor max at $x = c$.





Ex.1 Find the increasing and decreasing intervals and relative extrema for $f(x) = 10x^3 + 9x^2 + \frac{12}{5}x$.

⓪ Remember to keep in mind the domain. Here, domain is $(-\infty, \infty)$.

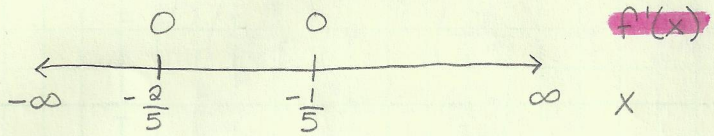
① Find the critical numbers for $f(x)$.

- $f'(x) = 30x^2 + 18x + \frac{12}{5}$ → defined everywhere
  $= \frac{1}{5}(150x^2 + 90x + 12)$
  $= \frac{6}{5}(25x^2 + 15x + 2)$
  $= \frac{6}{5}(5x+2)(5x+1)$

- $0 = \frac{6}{5}\underbrace{(5x+2)}_{=0}\underbrace{(5x+1)}_{=0}$
  $x = -\frac{2}{5}$ $\quad$ $x = -\frac{1}{5}$

- $x = -\frac{2}{5}$ and $x = -\frac{1}{5}$ are the critical numbers.

② Make a number line with the critical numbers and domain.



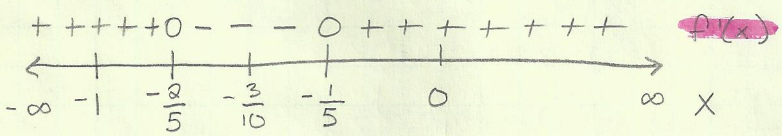
Pick x-values in each of the intervals on the number line.

- For $(-\infty, -\frac{2}{5})$, we can pick $x=-1$. (or $x=-2$, $x=-3$, etc.)
- For $(-\frac{2}{5}, -\frac{1}{5})$, we can pick $x=-\frac{3}{10}$. (or $x=-\frac{7}{20}$, $x=-\frac{13}{37}$, etc.)
- For $(-\frac{1}{5}, \infty)$, we can pick $x=0$. (or $x=1$, $x=2$, etc.)

Plug the x-values into the <u>derivative</u> $f'(x)$ to see if $f'(x)$ is positive or negative. We only care about positive or negative; the actual number does not matter.

- $f'(-1) = \frac{6}{5}(5(-1)+2)(5(-1)+1) = \frac{6}{5}(-3)(-4) = (+)(-)(-) = (+)$
- $f'(-\frac{3}{10}) = \frac{6}{5}(5(-\frac{3}{10})+2)(5(-\frac{3}{10})+1) = \frac{6}{5}(\frac{1}{2})(-\frac{1}{2}) = (+)(+)(-) = (-)$
- $f'(0) = \frac{6}{5}(5(0)+2)(5(0)+1) = \frac{6}{5}(2)(1) = (+)(+)(+) = (+)$

Since $f'(-1)$ is positive, $f'(x)$ is positive on the <u>entire</u> interval $(-\infty, -\frac{2}{5})$. Same for the other intervals.



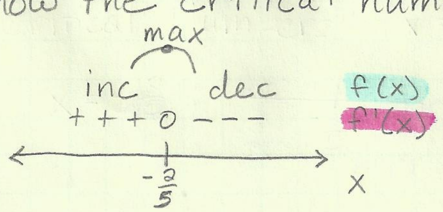
In LON-CAPA: No spaces!

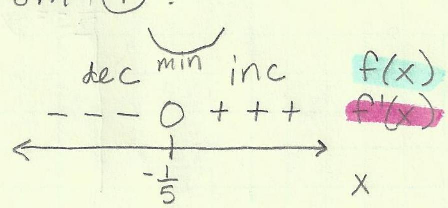
③ $f(x)$ is increasing where $f'(x)$ is positive: $\boxed{(-\infty,-\frac{2}{5})\cup(-\frac{1}{5},\infty)}$

$f(x)$ is decreasing where $f'(x)$ is negative: $\boxed{(-\frac{2}{5},-\frac{1}{5})}$

④ Use the 1st DT to find relative extrema.

Know the critical numbers from ①:





Rel. max at $x=-\frac{2}{5}$

Rel. min at $x=-\frac{1}{5}$

⑤ Plug the x-values for the relative extrema into the original function $f(x)$ to find the actual values of the extrema.

$$f\left(-\tfrac{2}{5}\right) = 10\left(-\tfrac{2}{5}\right)^3 + 9\left(-\tfrac{2}{5}\right)^2 + \left(\tfrac{12}{5}\right)\left(-\tfrac{2}{5}\right)$$
$$= \overset{5\cdot 2}{10}\left(\frac{-8}{\underset{5\cdot 25}{125}}\right) + 9\left(\frac{4}{25}\right) - \frac{24}{25}$$
$$= -\frac{16}{25} + \frac{36}{25} - \frac{24}{25}$$
$$= \frac{-4}{25}$$

$$\boxed{\text{Rel. max: } \left(-\tfrac{2}{5}, -\tfrac{4}{25}\right)}$$

$$f\left(-\tfrac{1}{5}\right) = 10\left(-\tfrac{1}{5}\right)^3 + 9\left(-\tfrac{1}{5}\right)^2 + \tfrac{12}{5}\left(-\tfrac{1}{5}\right)$$
$$= \overset{5\cdot 2}{10}\left(-\frac{1}{\underset{5\cdot 25}{125}}\right) + 9\left(\frac{1}{25}\right) - \frac{12}{25}$$
$$= -\frac{2}{25} + \frac{9}{25} - \frac{12}{25}$$
$$= -\frac{5}{25} = -\frac{1}{5}$$

$$\boxed{\text{Rel. min: } \left(-\tfrac{1}{5}, -\tfrac{1}{5}\right)}$$

Ex. 2 Find the relative extrema for $y = \sin(2x) + x$ on $(0, 2\pi)$.

⓪ They give us the domain/interval to care about: $(0, 2\pi)$.

① Find the critical numbers.

- $y' = \underbrace{\frac{d}{dx}(\sin(2x))} + \underbrace{\frac{d}{dx}(x)}_{=1}$

  Chain: Out $= \sin(x)$ In $= 2x$

  Out' $= \cos(x)$ In' $= 2$

  $\frac{d}{dx}(\sin(2x)) = \text{Out}'(\text{In}) \cdot \text{In}' = \cos(\text{In}) \cdot 2 = 2\cos(2x)$

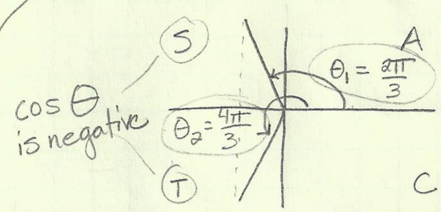
  $y' = 2\cos(2x) + 1$

- $0 = 2\cos(2x) + 1$ → Where does $\cos\theta = \frac{-1}{2}$?

$-2\cos(2x) = 1$

$\cos(2x) = -\frac{1}{2}$

$2x = \arccos\left(-\frac{1}{2}\right)$



$\frac{1}{2}\left(2x = \frac{2\pi}{3} + 2\pi n\right)\frac{1}{2}$

$x = \frac{\pi}{3} + \pi n$

$\frac{1}{2}\left(2x = \frac{4\pi}{3} + 2\pi n\right)\frac{1}{2}$

$x = \frac{2\pi}{3} + \pi n$

(+$2\pi n$ because we can go around the circle any number of times and get back to the same angle.)

- Now, add π to each value until the values are greater than 2π.

$x = \frac{\pi}{3}$

$x = \frac{\pi}{3} + \pi = \frac{4\pi}{3}$

$\cancel{x = \frac{4\pi}{3} + \pi = \frac{7\pi}{3} > 2\pi}$

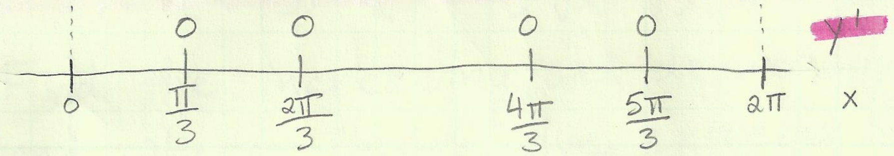
$x = \frac{2\pi}{3}$

$x = \frac{2\pi}{3} + \pi = \frac{5\pi}{3}$

$\cancel{x = \frac{5\pi}{3} + \pi = \frac{8\pi}{3} > 2\pi}$

- Critical numbers are $x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$.

② Make a number line for the interval with the critical numbers.

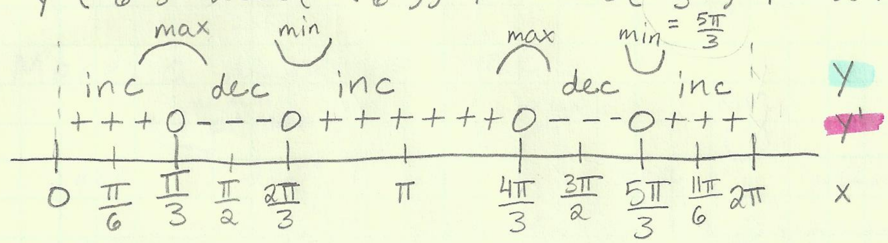


Pick x-values in each interval.

- For $(0, \frac{\pi}{3})$, $x = \frac{\pi}{6}$.
- For $(\frac{\pi}{3}, \frac{2\pi}{3})$, $x = \pi/2$.
- For $(\frac{2\pi}{3}, \frac{4\pi}{3})$, $x = \pi$.
- For $(\frac{4\pi}{3}, \frac{5\pi}{3})$, $x = \frac{3\pi}{2}$.
- For $(\frac{5\pi}{3}, 2\pi)$, $x = \frac{11\pi}{6}$.

Plug into $y' = 2\cos(2x) + 1$.

- $y'(\frac{\pi}{6}) = 2\cos(2(\frac{\pi}{6})) + 1 = 2\cos(\frac{\pi}{3}) + 1 = 2(\frac{1}{2}) + 1 = 2 > 0$ (+)
- $y'(\frac{\pi}{2}) = 2\cos(2(\frac{\pi}{2})) + 1 = 2\cos(\pi) + 1 = 2(-1) + 1 = -1 < 0$ (−)
- $y'(\pi) = 2\cos(2\pi) + 1 = 2(1) + 1 = 3 > 0$ (+)
- $y'(\frac{3\pi}{2}) = 2\cos(2(\frac{3\pi}{2})) + 1 = 2\cos(3\pi) + 1 = 2(-1) + 1 = -1 < 0$ (−)
- $y'(\frac{11\pi}{6}) = 2\cos(2(\frac{11\pi}{6})) + 1 = 2\cos(\underbrace{\frac{11\pi}{3}}_{= \frac{5\pi}{3}}) + 1 = 2(\frac{1}{2}) + 1 = 2 > 0$ (+)



③ Using the number line, by the 1st DT, we have relative maxima at $x = \frac{\pi}{3}$ and $x = \frac{4\pi}{3}$ and relative minima at $x = \frac{2\pi}{3}$ and $x = \frac{5\pi}{3}$.

④ Plug the x-values of the extrema into the original y equation to find the values of the min/max.

- $y(\frac{\pi}{3}) = \sin(2(\frac{\pi}{3})) + \frac{\pi}{3} = \sin(\frac{2\pi}{3}) + \frac{\pi}{3} = \frac{\sqrt{3}}{2} + \frac{\pi}{3}$
- $y(\frac{2\pi}{3}) = \sin(2(\frac{2\pi}{3})) + \frac{2\pi}{3} = \sin(\frac{4\pi}{3}) + \frac{2\pi}{3} = -\frac{\sqrt{3}}{2} + \frac{2\pi}{3}$
- $y(\frac{4\pi}{3}) = \sin(2(\frac{4\pi}{3})) + \frac{4\pi}{3} = \sin(\underbrace{\frac{8\pi}{3}}_{= \frac{2\pi}{3}}) + \frac{4\pi}{3} = \frac{\sqrt{3}}{2} + \frac{4\pi}{3}$
- $y(\frac{5\pi}{3}) = \sin(2(\frac{5\pi}{3})) + \frac{5\pi}{3} = \sin(\underbrace{\frac{10\pi}{3}}_{= \frac{4\pi}{3}}) + \frac{5\pi}{3} = -\frac{\sqrt{3}}{2} + \frac{5\pi}{3}$

$$\boxed{\begin{array}{ll} \text{Rel. max: } (\frac{\pi}{3}, \frac{\sqrt{3}}{2} + \frac{\pi}{3}) & \text{Rel. min: } (\frac{2\pi}{3}, -\frac{\sqrt{3}}{2} + \frac{2\pi}{3}) \\ \qquad (\frac{4\pi}{3}, \frac{\sqrt{3}}{2} + \frac{4\pi}{3}) & \qquad (\frac{5\pi}{3}, -\frac{\sqrt{3}}{2} + \frac{5\pi}{3}) \end{array}}$$

Ex. 3 The derivative of a polynomial is

$f'(x) = (x+3)^2(x+1)(x-1)(x-4)^3$.

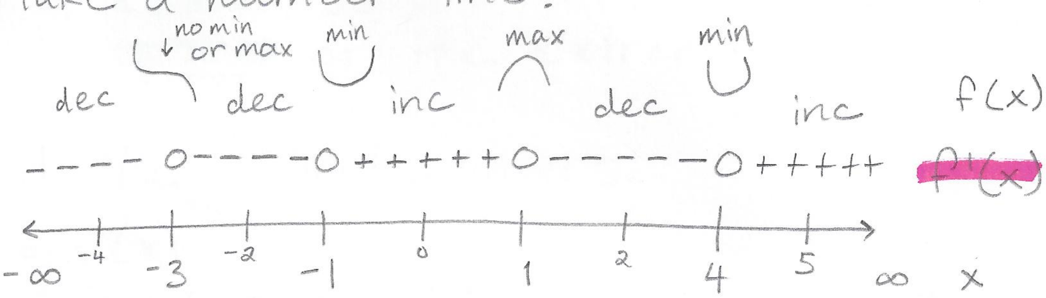
Find the increasing and decreasing intervals and where the relative extrema occur.

Note: Since we don't have the original function, we can't find the values of the relative extrema.

① Find the critical numbers.
Since we already have the first derivative, we don't have to differentiate anything. All we have to do is solve $f'(x)=0$.

$$0 = \underbrace{(x+3)^2}_{x=-3}\underbrace{(x+1)}_{x=-1}\underbrace{(x-1)}_{x=1}\underbrace{(x-4)^3}_{x=4}$$

② Make a number line.



Remember, we only care about the sign!

$$f'(-4) = (-4+3)^2(-4+1)(-4-1)(-4-4)^3$$
$$= (-)^2(-)(-)(-)^3$$
$$= (+)(-)(-)(-) = (-)$$

$$f'(-2) = (-2+3)^2(-2+1)(-2-1)(-2-4)^3$$
$$= (+)^2(-)(-)(-)^3$$
$$= (+)(-)(-)(-) = (-)$$

$$f'(0) = (0+3)^2(0+1)(0-1)(0-4)^3$$
$$= (+)^2(+)(-)(-)^3$$
$$= (+)(+)(-)(-) = (+)$$

$$f'(2) = (2+3)^2(2+1)(2-1)(2-4)^3$$
$$= (+)^2(+)(+)(-)^3$$
$$= (+)(+)(+)(-) = (-)$$

$$f'(5) = (5+3)^2(5+1)(5-1)(5-4)^3$$
$$= (+)^2(+)(+)(+)^3$$
$$= (+)(+)(+)(+) = (+)$$

③ Write intervals and extrema.

$\boxed{\begin{array}{l}\text{Inc: } (-1, 1) \cup (4, \infty) \\ \text{Dec: } \underbrace{(-\infty, -1)} \cup (1, 4)\end{array}}$

$\boxed{\begin{array}{l}\text{Rel. max of } f(x) \text{ @ } x = 1 \\ \text{Rel. min of } f(x) \text{ @ } x = -1, x = 4\end{array}}$

we had $(-\infty, -3) \cup (-3, -1)$, but since we don't switch from decreasing to increasing at $x = -3$, we can expand that interval to include $x = -3$.

Ex. 4 The derivative of a function is $h'(x) = xe^x(x^2 - 4x + 4)$.

Find the increasing and decreasing intervals and where the relative extrema occur.

Note: Again, we don't have the original function, so we can't find the values of the extrema.

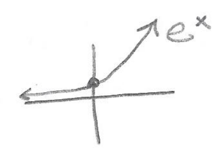
① Find the critical numbers.

$0 = h'(x)$

$0 = xe^x(x^2 - 4x + 4)$

$0 = \underset{x=0}{x}\, e^x \underbrace{(x-2)^2}_{x=2}$

$e^x > 0$ for all x, so this gives no zeros :



② Make a number line.

min   no min or max

dec   inc   inc

$- - - - - \; 0 \; + + + + + \; 0 \; + + + + + +$ $h'(x)$

$-\infty \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \quad \infty \quad x$

$h'(-1) = (-1)e^{-1}(-1-2)^2$
$= (-)(+)(-)^2$
$= (-)(+)(+) = (-)$

$h'(1) = (1)e^1(1-2)^2$
$= (+)(+)(-)^2$
$= (+)(+)(+) = (+)$

$h'(3) = (3)(e^3)(3-2)^2$
$= (+)(+)(+)^2 = (+)$

③ All for $h(x)$ → $\boxed{\begin{array}{l}\text{Inc: } (0, \infty) \\ \text{Dec: } (-\infty, 0)\end{array}}$ → Again, we expanded the interval because we didn't switch