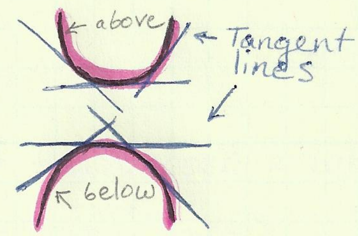
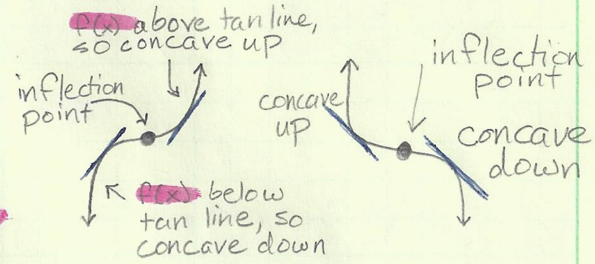
Lesson 19: Concavity; Inflection Points; Second Derivative Test



Def. concave up: where a function lies above its tangent lines; where $f''(x) > 0$

concave down: where a function lies below its tangent lines; where $f''(x) < 0$

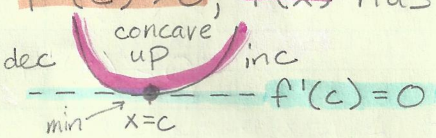
Def. $f(x)$ has an inflection point where $f(x)$ changes concavity (either up to down or down to up)
* Point is on $f(x)$, so the x-value must be in the domain of $f(x)$.



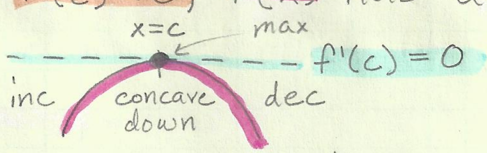
* Second Derivative Test (2nd DT)

Let $x=c$ be a critical number of $f(x)$ (i.e. $f'(x)=0$).

- If $f''(c) > 0$, $f(x)$ has a relative minimum at $x=c$.



- If $f''(c) < 0$, $f(x)$ has a relative maximum at $x=c$.



- If $f''(c) = 0$, 2nd DT is inconclusive, so use 1st DT.

(Note that for the above graphs, 1st DT also gives min/max.)

Ex. 1 Consider $f(x) = 3x^5 + 10x^4 - 2$.

(a) Find the intervals where $f(x)$ is concave up or concave down.

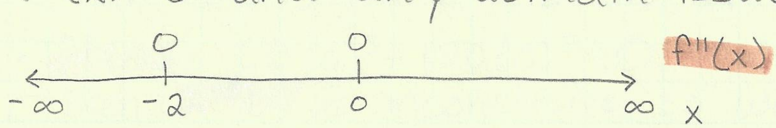
⓪ Domain: $(-\infty, \infty)$

① Find $f''(x)$. $\quad f'(x) = 15x^4 + 40x^3$
$f''(x) = 60x^3 + 120x^2$

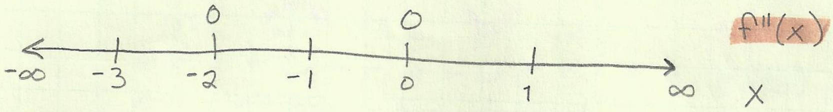
② Find where $f''(x) = 0$.

$$0 = 60x^3 + 120x^2$$
$$0 = \underbrace{60x^2}_{x=0}\ \underbrace{(x+2)}_{x=-2}$$

③ Make a number line with the x-values where $f''(x) = 0$ and any domain issues.

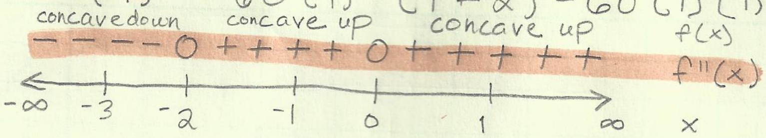


Pick x-values in each interval.



Find if $f''(x)$ is positive or negative at the chosen x-values.

- $f''(-3) = 60(-3)^2(-3+2) = 60(9)(-1) = (+)(+)(-) = (-)$
- $f''(-1) = 60(-1)^2(-1+2) = 60(1)(1) = (+)(+)(+) = (+)$
- $f''(1) = 60(1)^2(1+2) = 60(1)(1) = (+)(+)(+) = (+)$



④ Write the intervals.

Concave down ($f''(x) < 0$): $\boxed{(-\infty, -2)}$

Concave up ($f''(x) > 0$): $\boxed{(-2, \infty)}$

Since 0 is in the domain of f(x) and f(x) does not change concavity there, we can include it in the concave up interval

(b) Find the inflection point(s).

① Where does f(x) change concavity? $x = -2$.

② Is the x-value in the domain? Yes

③ Plug the x-value into f(x) to find the y-value of the inflection point.

$$\begin{aligned} f(-2) &= 3(-2)^5 + 10(-2)^4 - 2 \\ &= 3(-32) + 10(16) - 2 \\ &= -96 + 160 - 2 \\ &= 62 \end{aligned}$$

Inflection Point: $\boxed{(-2, 62)}$

(c) Find the relative extrema of f(x).

① Find the critical numbers of f(x). (where $f'(x) = 0$)

$0 = 15x^4 + 40x^3$

$0 = \underbrace{5x^3}_{x=0}\underbrace{(3x+8)}_{x=-8/3}$

$x = 0$ and $x = -\frac{8}{3}$ are the critical numbers of f(x).

② Use 2nd DT to determine where the min/max are. (We can either use the number line above, or we can plug the x-values into $f''(x)$.)

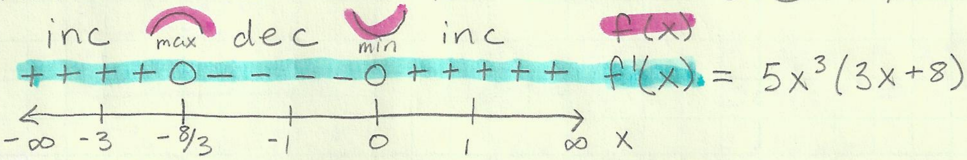
- $f''(-\frac{8}{3}) = 60(-\frac{8}{3})^3 + 120(-\frac{8}{3})^2 = -\frac{2560}{9} < 0$

  Then f(x) is concave down at $x = -\frac{8}{3}$ ∩, so rel. max at $x = -8/3$.

- $f''(0) = 60(0)^3 + 120(0)^2 = 0$

  Then 2nd DT is inconclusive, so we have to use 1st DT.

- Make number line for $f'(x)$ to see inc/dec intervals.



- $f'(-3) = 5(-3)^3(3(-3)+8) = (+)(-)(-9+8) = (+)(-)(-) = (+)$
  $f'(-1) = 5(-1)^3(3(-1)+8) = (+)(-)(-3+8) = (+)(-)(+) = (-)$
  $f'(1) = 5(1)^3(3(1)+8) = (+)(+)(+) = (+)$

Rel. max at $x = -\frac{8}{3}$
Rel. min at $x = 0$

③ Use $f(x)$ to find the y-values of the relative extrema.

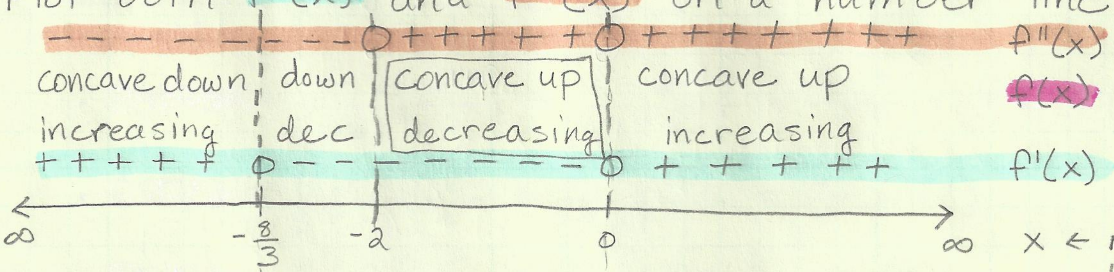
$f(-\frac{8}{3}) = 3(-\frac{8}{3})^5 + 10(-\frac{8}{3})^4 - 2 = \frac{8030}{81}$

$f(0) = 3(0)^5 + 10(0)^4 - 2 = -2$

$$\boxed{\begin{array}{ll}\text{Rel. max:} & \left(-\frac{8}{3}, \frac{8030}{81}\right) \\ \text{Rel. min:} & (0, -2)\end{array}}$$

(d) Find the intervals where $f(x)$ is concave up and decreasing.

Plot both $f'(x)$ and $f''(x)$ on a number line:



← mark all x-values where $f'(x) = 0$ OR $f''(x) = 0$

$f(x)$ is concave up and decreasing $\boxed{(-2, 0)}$

<u>Ex. 2</u> Consider $f(x) = \ln(x^2+2)$.
(a) Find where $f(x)$ is concave up or down.

⓪ Domain: Can only take natural log of numbers >0.
$x^2 + 2 > 0$ for all x-values, so <u>no</u> domain issues.
$(-\infty, \infty)$

① Find $f''(x)$.

$f'(x) = \frac{d}{dx}(\ln(x^2+2))$

<u>Chain</u>: Out $= \ln(x)$ In $= x^2+2$
Out$' = \frac{1}{x}$ In$' = 2x$

$f'(x) = \frac{1}{x^2+2} \cdot (2x) = \frac{2x}{x^2+2}$

$$f''(x) = \frac{d}{dx}\underbrace{\left(\frac{2x}{x^2+2}\right)}$$

Quotient Rule: Top $= 2x$ Top' $= 2$ Bottom $= x^2+2$ Bottom' $= 2x$

$$f''(x) = \frac{T' \cdot B - T \cdot B'}{B^2} = \frac{2(x^2+2) - 2x(2x)}{(x^2+2)^2}$$

$$= \frac{2x^2+4-4x^2}{(x^2+2)^2}$$

$$= \frac{4-2x^2}{(x^2+2)^2}$$

② Find where $f''(x) = 0$

$$0 = \frac{4-2x^2}{(x^2+2)^2}$$ ← Denominator never equals 0. A fraction can only equal 0 when the numerator equals 0.

$$0 = 4-2x^2$$
$$2x^2 = 4$$
$$x^2 = 2$$
$$x = \pm\sqrt{2} \approx 1.414$$

③ Make a number line.

down up down $f(x)$

$- - - 0 + + + + + + 0 - - -$ $f''(x)$

$-\infty$ -2 $-\sqrt{2}$ 0 $\sqrt{2}$ 2 ∞ x

- $f''(-2) = \dfrac{4-2(-2)^2}{((-2)^2+2)^2} = \dfrac{4-2(4)}{(4+2)^2} = \dfrac{4-8}{6^2} = \dfrac{(-)}{(+)} = (-)$
- $f''(0) = \dfrac{4-2(0)^2}{(0^2+2)^2} = \dfrac{4-2(0)}{(2)^2} = \dfrac{4}{4} = (+)$
- $f''(2) = \dfrac{4-2(2)^2}{((2)^2+2)^2} = \dfrac{4-8}{(4+2)^2} = \dfrac{(-)}{(+)} = (-)$

④ Write intervals.

$$\boxed{\begin{array}{l}\text{Concave Down: } (-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty) \\ \text{Concave Up: } (-\sqrt{2}, \sqrt{2})\end{array}}$$

(b) Find the inflection point(s) for $f(x)$.

① Where does $f(x)$ change concavity? $x = -\sqrt{2}$, $x = \sqrt{2}$

② Are the x-values in the domain? Yes

③ Find the y-values of the inflection point(s) using $f(x)$.

$f(-\sqrt{2}) = \ln((-\sqrt{2})^2+2) = \ln(2+2) = \ln(4)$

$f(\sqrt{2}) = \ln((\sqrt{2})^2+2) = \ln(2+2) = \ln(4)$

Inflection Points: $\boxed{\begin{array}{l}(-\sqrt{2}, \ln(4)) \\ (\sqrt{2}, \ln(4))\end{array}}$

Ex. 3 Consider $f(x) = (x^2 - 4x + 1)e^x$.

(a) Find where $f(x)$ is concave up or down.

⓪ Domain: $(-\infty, \infty)$

① Find $f''(x)$.

$$f'(x) = \frac{d}{dx}\Big(\underbrace{(x^2-4x+1)\,e^x}_{\text{product rule}}\Big)$$

$$= \frac{d}{dx}(x^2-4x+1)\cdot e^x + (x^2-4x+1)\cdot\frac{d}{dx}(e^x)$$

$$= (2x-4)e^x + (x^2-4x+1)e^x$$

$$= e^x[2x-4+(x^2-4x+1)]$$

factor out e^x to make finding $f''(x)$ easier

$$= \underbrace{e^x\,(x^2-2x-3)}_{\text{product again}}$$

$$f''(x) = \frac{d}{dx}(e^x)\cdot(x^2-2x-3) + e^x\cdot\frac{d}{dx}(x^2-2x-3)$$

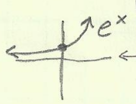
$$= e^x(x^2-2x-3) + e^x(2x-2)$$

$$= e^x[(x^2-2x-3)+(2x-2)]$$

$$= e^x(x^2-5)$$

② Find where $f''(x) = 0$.

$$0 = e^x\,\underbrace{(x^2-5)}$$



← always >0

$x^2 - 5 = 0$

$x^2 = 5$

$x = \pm\sqrt{5} \approx \pm 2.236$

③ Make a number line.

up down up $f(x)$

$+ + + + \;0\; - - - - \;0\; + + + +$ $f''(x)$

$-\infty \quad -3 \quad -\sqrt{5} \quad 0 \quad \sqrt{5} \quad 3 \quad \infty \quad x$

- $f''(-3) = e^{-3}((-3)^2-5) = e^{-3}(9-5) = e^{-3}(4) > 0$ (+)
- $f''(0) = e^0(0^2-5) = (1)(-5) = -5 < 0$ (−)
- $f''(3) = e^3(3^2-5) = e^3(9-5) = e^3(4) > 0$ (+)

④ Write intervals.

$$\boxed{\begin{array}{l}\text{Concave Down: } (-\sqrt{5}, \sqrt{5}) \\ \text{Concave Up: } (-\infty, -\sqrt{5}) \cup (\sqrt{5}, \infty)\end{array}}$$

(6) Find the inflection point(s) for $f(x)$.

① Where does $f(x)$ change concavity? $x = -\sqrt{5}$ and $x = \sqrt{5}$

② Are the x-values in the domain? Yes

③ Find the y-values of the inflection points using $f(x)$.

$$
\begin{aligned}
f(-\sqrt{5}) &= \left((-\sqrt{5})^2 - 4(-\sqrt{5}) + 1\right) e^{-\sqrt{5}} \\
&= (5 + 4\sqrt{5} + 1) e^{-\sqrt{5}} \\
&= (6 + 4\sqrt{5}) e^{-\sqrt{5}}
\end{aligned}
$$

$$
\begin{aligned}
f(\sqrt{5}) &= \left((\sqrt{5})^2 - 4(\sqrt{5}) + 1\right) e^{\sqrt{5}} \\
&= (5 - 4\sqrt{5} + 1) e^{\sqrt{5}} \\
&= (6 - 4\sqrt{5}) e^{\sqrt{5}}
\end{aligned}
$$

Inflection Points: $\left(-\sqrt{5},\ (6+4\sqrt{5})e^{-\sqrt{5}}\right)$, $\left(\sqrt{5},\ (6-4\sqrt{5})e^{\sqrt{5}}\right)$