* Lesson 20: Absolute Extrema on an Interval

Ex.1 Find and classify the relative extrema of $f(x) = x^2$.

1 Find the critical numbers of $f(x)$.

$$f'(x) = 2x$$
$$0 = 2x$$
$$0 = x$$

2 Use 1st or 2nd DT to check if we have a rel min/max at $x = 0$.

Here, 2nd DT is easier than making the number line for the 1st DT.

$f''(x) = 2 > 0$, so $f(x)$ is always concave up ∪, which means there is a relative min at $x = 0$.

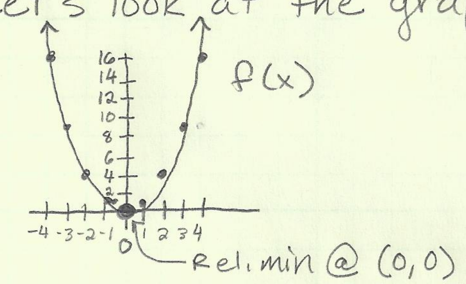
3 Use $f(x)$ to find the value of the relative extrema.

$f(0) = 0^2 = 0$

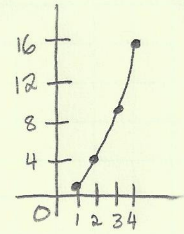
$\boxed{\text{Relative min : } (0,0)}$

Ex.2 Find the absolute extrema of $f(x) = x^2$ on the closed interval $[1, 4]$.

Let's look at the graphs first.



On the interval $[1, 4]$, we have this part of the graph:



Comparing the two graphs, the left graph shows the relative minimum at $(0, 0)$, but the right graph does not include the point $(0,0)$.
However, because we have a closed interval $[1,4]$, we will have both an absolute min and an absolute max. This will be true for any closed interval.

To find the absolute extrema of $f(x)$ on the interval $[a,b]$:

1 Find all critical numbers of $f(x)$.
2 Evaluate $f(x)$ at all critical numbers in the interval $[a,b]$ AND at the end points $x = a$ and $x = b$.
3 Compare the $f(x)$ values from 2. The largest is the absolute max, and the smallest is the absolute min.

① Find the critical numbers.

$$f'(x) = 2x$$
$$0 = 2x$$
$$0 = x$$

② $x=0$ is not in the interval $[1,4]$, so we don't check the value of $f(x)$ there. This means we only have to evaluate $f(x)$ at $x=1$ and $x=4$.

x	$f(x) = x^2$	
1	$1^2 = 1$	← smaller
4	$4^2 = 16$	← bigger

③ $\boxed{\begin{array}{l}\text{Absolute Max: } (4,16) \\ \text{Absolute Min: } (1,1)\end{array}}$

For the following, find the absolute extrema on the given closed interval.

Ex. 3 $f(x) = x^2 + 3x - 1$ on $[-4, 2]$

① $f'(x) = 2x + 3$
$$0 = 2x + 3$$
$$-3 = 2x$$
$$-\tfrac{3}{2} = x$$

②

x	$f(x) = x^2 + 3x - 1$	
-4	$(-4)^2 + 3(-4) - 1 = 16 - 12 - 1 = 3$	
$-\frac{3}{2}$	$(-\frac{3}{2})^2 + 3(-\frac{3}{2}) - 1 = \frac{9}{4} - \frac{9}{2} - 1 = \frac{9}{4} - \frac{18}{4} - \frac{4}{4} = -\frac{13}{4}$	← smallest
2	$(2)^2 + 3(2) - 1 = 4 + 6 - 1 = 9$	← biggest

③ $\boxed{\begin{array}{l}\text{Abs. Max: } (2,9) \\ \text{Abs. Min: } (-\frac{3}{2}, -\frac{13}{4})\end{array}}$

Ex. 4 $f(x) = \underline{2x}\,\underline{e^{-3x}} + 1$ on $[0, 1]$

① Product Rule: $f'(x) = \underbrace{\frac{d}{dx}(2x)}_{=2} e^{-3x} + 2x \underbrace{\frac{d}{dx}(e^{-3x})}_{\text{Chain}} + \underbrace{\frac{d}{dx}(1)}_{=0}$

Chain:
Out $= e^x$ In $= -3x$
Out' $= e^x$ In' $= -3$
$= -3e^{-3x}$

$$f'(x) = 2e^{-3x} + 2x(-3e^{-3x}) + 0$$
$$= 2e^{-3x} - 6xe^{-3x}$$
$$= \underbrace{2e^{-3x}}_{\text{never } = 0} \underbrace{(1 - 3x)}_{\substack{1-3x=0 \\ x = \frac{1}{3}}}$$

$x = \frac{1}{3}$ is the only critical number.

②

x	$f(x) = 2xe^{-3x}+1$
0	$2(0)e^{-3(0)}+1 = 0+1 = 1$ ← smallest
$\frac{1}{3}$	$2(\frac{1}{3})e^{-3(1/3)}+1 = \frac{2}{3}e^{-1}+1 \approx 1.245$ ← biggest
1	$2(1)e^{-3(1)}+1 = 2e^{-3}+1 \approx 1.0996$

③ $\boxed{\text{Abs Max: } (\frac{1}{3}, \frac{2}{3}e^{-1}+1) \quad \text{Abs Min: } (0,1)}$

Ex. 5 $y = \frac{1}{2x^2+1}$ on $[-1, 2]$

① Can find y' as written using the quotient rule, or rewrite as $y = (2x^2+1)^{-1}$ and find y' using the chain rule.

$y = (2x^2+1)^{-1}$ Chain: $Out = x^{-1}$, $Out' = -x^{-2}$; $In = 2x^2+1$, $In' = 4x$

$$y' = -(2x^2+1)^{-2}(4x)$$
$$= -\frac{4x}{(2x^2+1)^2}$$

$0 = -\frac{4x}{(2x^2+1)^2}$ → A fraction can only equal 0 if the numerator equals 0.

$$0 = -4x$$
$$0 = x$$

②

x	$f(x) = \frac{1}{2x^2+1}$
-1	$\frac{1}{2(-1)^2+1} = \frac{1}{2+1} = \frac{1}{3}$
0	$\frac{1}{2(0)^2+1} = \frac{1}{1} = 1$ ← biggest
2	$\frac{1}{2(2)^2+1} = \frac{1}{8+1} = \frac{1}{9}$ ← smallest

③ $\boxed{\text{Abs. Max: } (0,1) \quad \text{Abs. Min: } (2, \frac{1}{9})}$

Ex. 6 $h(x) = 3x^5 e^x + 1$ on $[-6, 1]$

① Product Rule: $h'(x) = \frac{d}{dx}(3x^5)e^x + 3x^5\frac{d}{dx}(e^x) + \frac{d}{dx}(1)$
$$= 15x^4e^x + 3x^5e^x + 0$$
$$= \underbrace{3}_{\neq 0}\underbrace{x^4}_{x=0}\underbrace{e^x}_{\neq 0}\underbrace{(5+x)}_{x=-5}$$

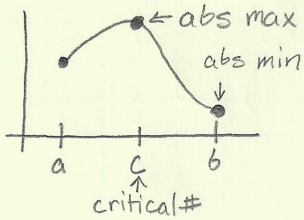
→ $x=0$ and $x=-5$ are the critical numbers.

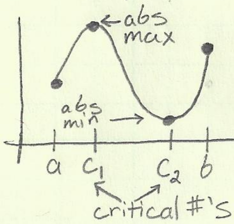
②

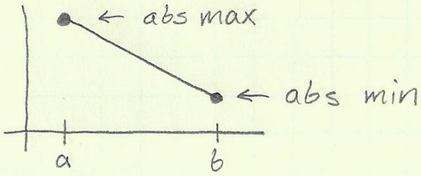
x	$h(x) = 3x^5e^x+1$
-6	$3(-6)^5e^{-6}+1 = -23328e^{-6}+1 \approx -56.8$
-5	$3(-5)^5e^{-5}+1 = -9375e^{-5}+1 \approx -62.2$
0	$3(0)^5e^0+1 = 0+1 = 1$
1	$3(1)^5e^1+1 = 3e+1 \approx 9.2$

③ $\boxed{\text{Abs Max: } (1, 3e+1) \quad \text{Abs Min: } (-5, -9375e^{-5}+1)}$

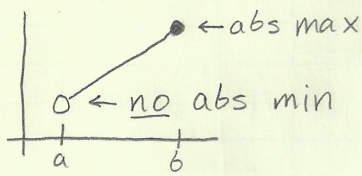
We call the interval $[a, b]$ closed because both the endpoints $x=a$ and $x=b$ are included.
When the interval is closed, there is always an absolute max <u>and</u> an absolute min.
Some ways this can happen:

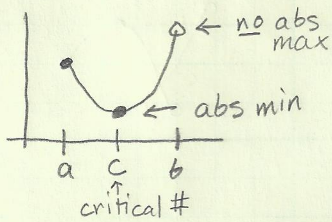


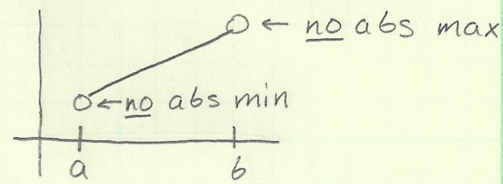




For intervals that aren't closed, i.e. $[a, b)$, $(a, b]$, or (a, b), $f(x)$ may have an absolute min, absolute max, both, or neither. For instance,







On intervals that aren't closed, you will almost always be told if you have to find the absolute min or max.

<u>Ex.7</u> Find the $\underbrace{\text{absolute max}}_{\text{looking for the largest y-value}}$ of $y = x^3 + x^2 + 2$ on $(-1, 0)$.

① $y' = 3x^2 + 2x$
$0 = \underbrace{x}_{x=0}\underbrace{(3x+2)}_{x=-\frac{2}{3}}$ are the critical numbers.

② $x=0$ and $x=1$ are not included in the interval, so we don't check either end point.
$x=0$ is also a critical number, but again, it's not included in $(-1, 0)$.
This means the only value we have to check is $f(-\frac{2}{3})$.

$$f(-\tfrac{2}{3}) = (-\tfrac{2}{3})^3 + (-\tfrac{2}{3})^2 + 2$$
$$= -\frac{8}{27} + \frac{4}{9} + 2$$
$$= -\frac{8}{27} + \frac{12}{27} + \frac{54}{27}$$
$$= \frac{58}{27}$$

③ $\boxed{\text{Abs Max: } \left(-\frac{2}{3}, \frac{58}{27}\right)}$

Ex.8 Find the absolute minimum of $y = \frac{x^2}{x+2}$ on $(-2, 3]$.

Note: When $x=-2$, y is undefined because the denominator is 0. Moreover, we have a vertical asymptote at $x=-2$.
This means that $x=-2$ is not in the domain for $y = \frac{x^2}{x+2}$, so it should not be included in any interval we write.

① Quotient Rule: Top $= x^2$ Bottom $= x+2$
Top$'$ $= 2x$ Bottom$'$ $= 1$

$$y' = \frac{\text{Top}' \cdot \text{Bottom} - \text{Top} \cdot \text{Bottom}'}{\text{Bottom}^2}$$

$$= \frac{2x(x+2) - x^2(1)}{(x+2)^2}$$

$$= \frac{2x^2 + 4x - x^2}{(x+2)^2}$$

$$0 = \frac{x^2+4x}{(x+2)^2}$$ → Only $= 0$ if numerator $= 0$.

$$0 = x^2 + 4x$$
$$0 = x(x+4)$$

$x=0$ $x=-4$ are the critical numbers.

② $x=-4$ is not in the interval $(-2, 3]$, and $x=-2$ is not included in the interval.
We still have to check at $x=0$ and $x=3$.

x	$y = \frac{x^2}{x+2}$
0	$\frac{0^2}{0+2} = \frac{0}{2} = 0$
3	$\frac{3^2}{3+2} = \frac{9}{5}$

← We're looking for the abs min, so we're looking for the smaller value.

③ $\boxed{\text{Abs Min: } (0, 0)}$