Lesson 21: Graphical Interpretation of Derivatives

Given the **graph of $f'(x)$**, we can find where $f(x)$ is or has the following:

1. critical numbers: where the graph *touches* the x-axis (i.e. $f'(x)$ is zero)

2. increasing: where the graph is *above* the x-axis (i.e. $f'(x)$ is positive)

3. decreasing: where the graph is *below* the x-axis (i.e. $f'(x)$ is negative)

4. relative max: where the graph *crosses* the x-axis from *positive* to *negative*

   (i.e. $f(x)$ goes from increasing to decreasing)

5. relative min: where the graph *crosses* the x-axis from *negative* to *positive*

   (i.e. $f(x)$ goes from decreasing to increasing)

6. concave up: where $f'(x)$ is *increasing* tan line has pos slope

   (i.e. where the tangent lines to the graph have positive slope)

   (i.e. the derivative of $f'(x)$ is positive, so $f''(x)>0$)

   ↳ since $f''(x)$ is the derivative of $f'(x)$

7. concave down: where $f'(x)$ is *decreasing*

   (i.e. where the tangent lines to the graph have negative slope)

   (i.e. the derivative of $f'(x)$ is negative, so $f''(x)<0$)

8. inflection point: where $f'(x)$ has a *horizontal tangent* AND

   $f'(x)$ changes from increasing to decreasing *or* decreasing to increasing

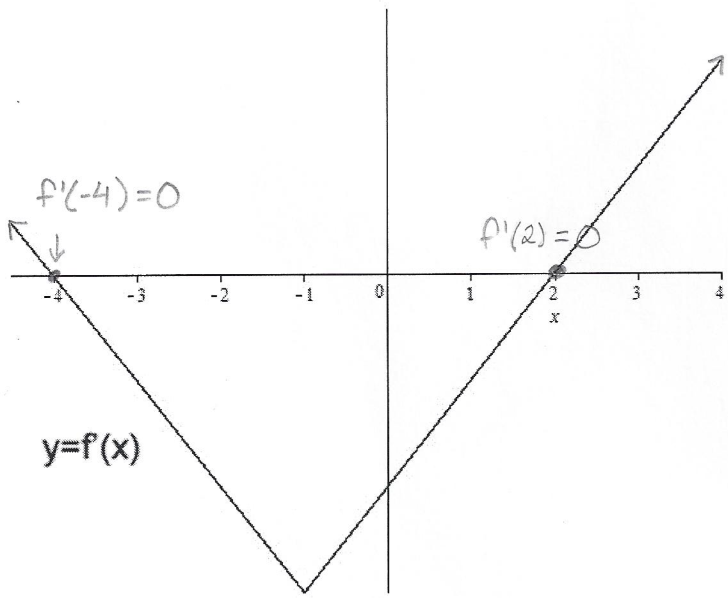
   (i.e. where $f''(x)$ changes sign)

✳ **Note:** All of these are x-values (including the intervals).

- We need the equation or graph of $f(x)$ to find the y-values.

Given the following **graphs of $f'(x)$**, find the requested information about $f(x)$.

Ex. 1



Critical Numbers:

$x = -4, 2$

Increasing Intervals: $f'(x) > 0$

$(-\infty, -4) \cup (2, \infty)$

Decreasing Intervals: $f'(x) < 0$

$(-4, 2)$

Relative Maxima: pos → neg

at $x = -4$

Relative Minima: neg → pos

at $x = 2$

Concave Up: $f'(x)$ increasing

$(-1, \infty)$

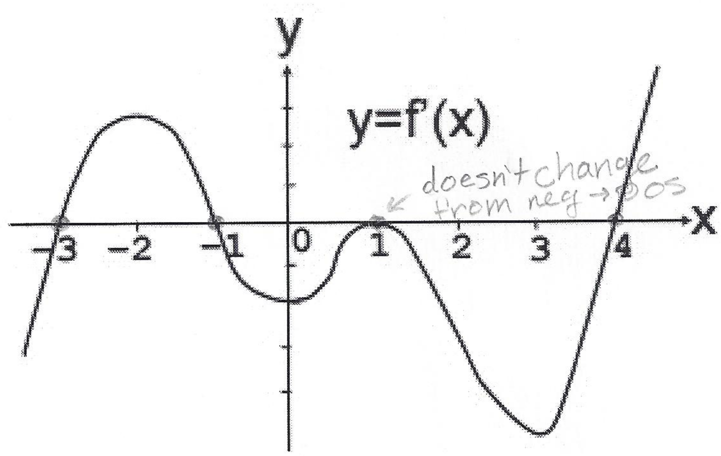
Concave Down: $f'(x)$ decreasing

$(-\infty, -1)$

Inflection Points:

at $x = -1$

Ex. 2



Critical Numbers:

$x = -3, -1, 1, 4$

Increasing Intervals:

$(-3, -1) \cup (4, \infty)$

Decreasing Intervals:

$(-\infty, -3) \cup (-1, 4)$

Relative Maxima:

at $x = -1$

Relative Minima:

at $x = -3, 4$

Concave Up:

$(-\infty, -2) \cup (0, 1) \cup (3, \infty)$

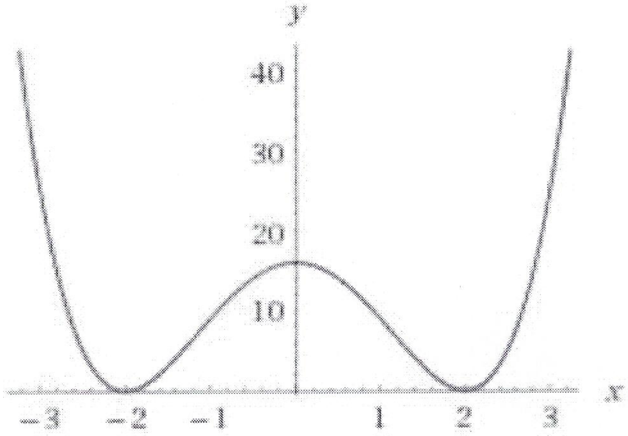
Concave Down:

$(-2, 0) \cup (1, 3)$

Inflection Points:

at $x = -2, 0, 1, 3$

Ex. 3 The following graph is of $f'(x)$. Choose the correct statement(s) about $f(x)$.



I. On $(-2, 2)$, $f(x)$ is increasing.

II. On $(-\infty, -2)$, $f(x)$ is concave up.

III. $f(x)$ has a relative maximum at $x = 0$.

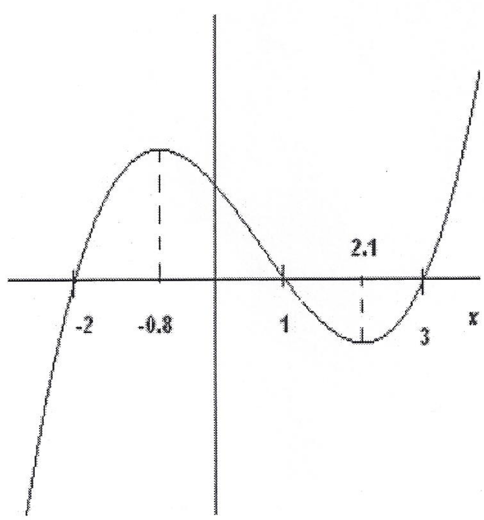
✓ I. On $(-2,2)$, $f'(x) > 0$, so $f(x)$ is increasing.
Note: $f(x)$ may be increasing on other intervals, but the statement is for this interval, not only this interval.

X II. On $(-\infty, -2)$, $f'(x)$ is decreasing, so its derivative $f''(x)$ is negative, which means $f(x)$ is concave down.

X III. At $x=0$, $f'(x) \neq 0$, so $f(x)$ doesn't even have a critical number there.
Note: At $x=0$, we have a horizontal tangent and $f'(x)$ switches inc → dec, so $f(x)$ actually has an inflection point there.

Only I is true.

Ex. 4 The graph of the **derivative** of a function $f(x)$ is shown below. Choose the correct statement(s) regarding $f(x)$.



I. On $(2.1, \infty)$, $f(x)$ is increasing.

II. On $(-0.8, 2.1)$, $f(x)$ is concave down.

III. $f(x)$ has a relative maximum at $x = -0.8$.

X I. On $(2.1, \infty)$, $f'(x)$ is increasing, so $f(x)$ is concave up.

✓ II. On $(-0.8, 2.1)$, $f'(x)$ is decreasing, so $f(x)$ is concave down.

X III. At $x = -0.8$, $f'(x) \neq 0$, so $f(x)$ can't have a relative extremum at $x = -0.8$.
Note: $f'(x)$ switches from inc. to dec., so $f(x)$ has an inflection point at $x = -0.8$.

Only II is true.

For instance, 

Ex. 5 Let $f(x)$ be a polynomial whose derivative is always increasing. Choose the correct statement(s).

I. $f(x)$ has an inflection point.
II. $f(x)$ has a relative maximum.
III. $f(x)$ is always concave up.

- $f'(x)$ is increasing on $(-\infty, \infty)$
- $f''(x) > 0$ on $(-\infty, \infty)$
- $f(x)$ concave up on $(-\infty, \infty)$

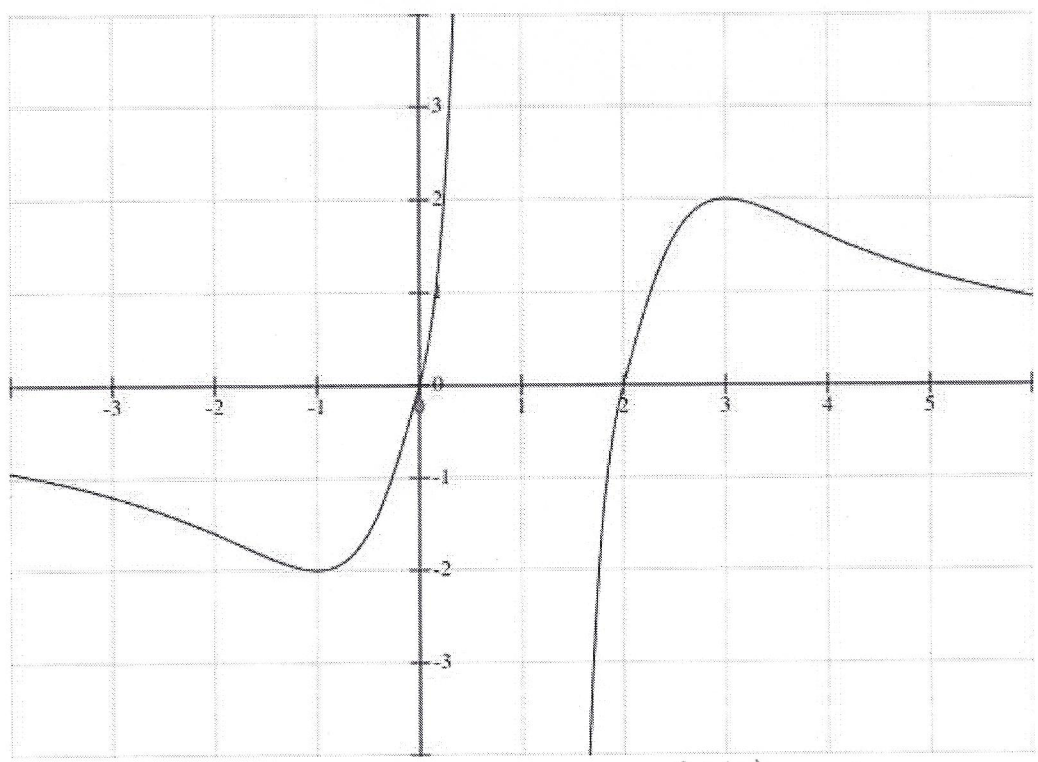
X I. $f(x)$ never changes concavity, so $f(x)$ has no inflection points.

X II. By 2nd DT, $f(x)$ being concave up ∪ means $f(x)$ could possibly have a min, but it definitely does not have have a max.
By 1st DT, since $f'(x)$ is always increasing, $f'(x)$ can never go from positive to negative, so $f(x)$ can never go from increasing to decreasing, so $f(x)$ never has a max.

✓ III. From the above, we see $f''(x) > 0$ always.

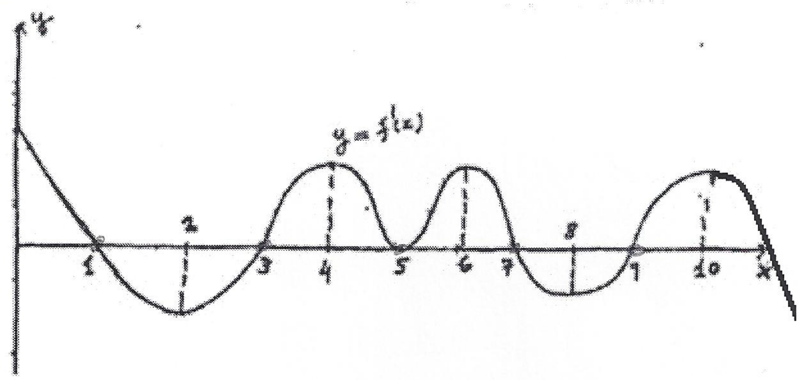
Only III is true.

Ex. 6 Below is the graph of the **derivative** of a function $f(x)$. On what interval is $f(x)$ both increasing and concave down?



$f(x)$ is increasing when $f'(x) > 0$.
$f(x)$ is concave down when $f'(x)$ is decreasing.
The only interval where the above graph is positive and decreasing is $\boxed{(3, \infty)}$.

Ex. 7



Critical Numbers:

$x = 1, 3, 5, 7, 9$ (and possibly 11, but the x-value is not marked there)

Increasing Intervals:

$(-\infty, 1) \cup (3, 7) \cup (9, 11)$

Decreasing Intervals:

$(1, 3) \cup (7, 9) \cup (11, \infty)$

Relative Maxima:

$x = 1, 7, 11$

Relative Minima:

$x = 3, 9$

Concave Up:

$(2, 4) \cup (5, 6) \cup (8, 10)$

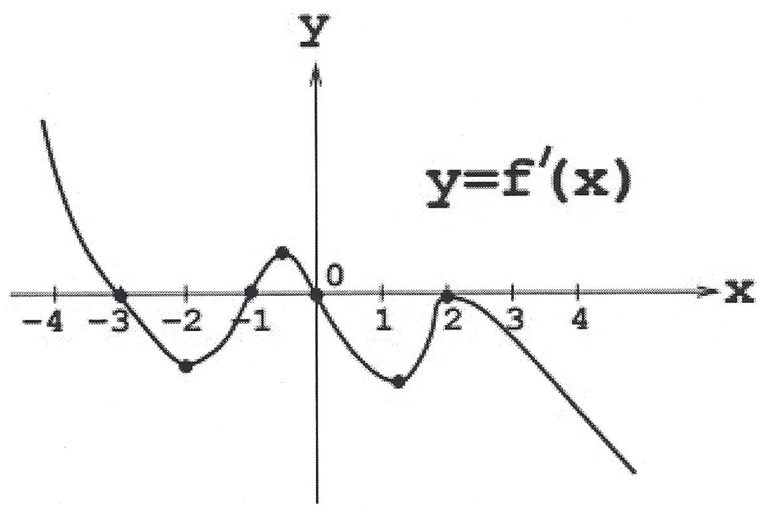
Concave Down:

$(-\infty, 2) \cup (4, 5) \cup (6, 8) \cup (10, \infty)$

Inflection Points:

$x = 2, 4, 5, 6, 8, 10$

Ex. 8



Critical Numbers:

$x = -3, -1, 0, 2$

Increasing Intervals:

$(-\infty, -3) \cup (-1, 0)$

Decreasing Intervals:

$(-3, -1) \cup (0, \infty)$

Relative Maxima:

$x = -3, 0$

Relative Minima:

$x = -1$

Concave Up:

$(-2, -\frac{1}{2}) \cup (1, 2)$

Concave Down:

$(-\infty, -2) \cup (-\frac{1}{2}, 1) \cup (2, \infty)$

Inflection Points:

$x = -2, -\frac{1}{2}, 1, 2$