# Lesson 22: Limits at Infinity

A "limit at infinity" means that $x \to \infty$ or $x \to -\infty$.
The actual limit can be 0, $\infty$, $-\infty$, or any number.

**Ex. 1** Find $\lim_{x\to\infty} -\frac{3}{x}$.

Since the only $x$ is in the denominator, we only have think about what happens when the denominator gets really big. As we divide by bigger and bigger numbers, the whole fraction gets smaller and smaller.
That means that as $x$ goes to infinity, the whole fraction goes to zero, so $\lim_{x\to\infty} -\frac{3}{x} = \boxed{0}$.

* **Note:** From earlier limit lessons, we can write

$$\lim_{x\to\infty} -\frac{3}{x} = -3\cdot\left(\lim_{x\to\infty}\frac{1}{x}\right) = -3\cdot 0 = 0.$$

**Ex. 2** Find $\lim_{x\to-\infty} \frac{x}{2}$.

Since the only $x$ is in the numerator, we only have to think about what happens when the numerator gets really big. $x \to -\infty$, so all the numbers are really big and negative. Dividing by 2 does not change $x$ really big or $x$ being negative, so $\lim_{x\to-\infty}\frac{x}{2} = \boxed{-\infty}$.

* **Note:** Again, we could write $\lim_{x\to-\infty}\frac{x}{2} = \frac{1}{2}\lim_{x\to-\infty} x = \frac{1}{2}(-\infty) = -\infty$.

**Ex. 3** Find $\lim_{x\to-\infty}\left(\frac{1}{x} - \frac{x^2}{5}\right)$.

Now, we have an $x$ in a numerator and denominator, but we can consider the fractions separately first.
$\frac{1}{x}$ has an $x$ only in the denominator, so $\lim_{x\to-\infty}\frac{1}{x} = 0$
$\frac{x^2}{5}$ has an $x$ only in the numerator, so we know when $x$ gets really big, the whole fraction gets big. Since $x \to -\infty$, the numbers will be negative, BUT we have $x^2$, so the numerator will be positive and getting big, so $\lim_{x\to-\infty}\frac{x^2}{5} = \infty$

Now, we have $\lim_{x\to-\infty}\left(\frac{1}{x} - \frac{x^2}{5}\right) = \lim_{x\to-\infty}\left(\frac{1}{x}\right) - \lim_{x\to-\infty}\frac{x^2}{5}$

$$= 0 - \infty$$
$$= \boxed{-\infty}$$

Ex. 4 Find $\lim\limits_{x\to\infty} \dfrac{1+2x-x^2}{2x-3}$.

We have $x$ in the numerator and denominator of a single fraction this time.

Numerator: $\lim\limits_{x\to\infty}(1+2x-x^2) = ?$

We may try to write

$$\cancel{\begin{aligned}\lim_{x\to\infty}(1+2x-x^2) &= \lim_{x\to\infty}(1) + 2\lim_{x\to\infty}(x) - \lim_{x\to\infty}(x^2)\\ &= 1 + 2\cdot\infty - \infty\\ &= 1+\infty = \infty\end{aligned}}$$

but this not true!

$x^2$ will go to infinity much faster than $x$.

We can't treat $\infty$ like a regular number. Instead, we need to think about the "dominating term".
For our class, the dominating term is always the $x$ with the highest power along with its coefficient.

This means
$$\begin{aligned}\lim_{x\to\infty}(1+2x-x^2) &= \lim_{x\to\infty}(-x^2)\\ &= -\lim_{x\to\infty} x^2\\ &= -\infty\end{aligned}$$

Denominator: We have to treat the denominator similarly, so
$$\begin{aligned}\lim_{x\to\infty}(2x-3) &= \lim_{x\to\infty}(2x)\\ &= 2\lim_{x\to\infty}(x)\\ &= \infty\end{aligned}$$

Now for the whole fraction, we have
$$\lim_{x\to\infty}\frac{1+2x-x^2}{2x-3} = \frac{-\infty}{\infty}.$$
Again, $\infty$ is not a normal number, so we can't just cancel them. Instead, we want to know if the numerator or denominator goes to $\infty$ faster, so again, we can look at the dominating terms.
$$\begin{aligned}\lim_{x\to\infty}\frac{1+2x-x^2}{2x-3} &= \lim_{x\to\infty}\frac{-x^2}{2x}\\ &= \lim_{x\to\infty}\frac{-x}{2}\\ &= -\frac{1}{2}\left(\lim_{x\to\infty} x\right)\\ &= \boxed{-\infty}\end{aligned}$$

**Ex. 5** Find $\lim_{x \to -\infty} \frac{x^2+x+1}{1-3x^2}$.

We identify the dominating terms for the numerator and denominator, then look at the fraction.

$$\lim_{x \to -\infty} \frac{x^2+x+1}{1-3x^2} = \lim_{x \to -\infty} \frac{x^2}{-3x^2} = \lim_{x \to -\infty} \frac{1}{-3} = \boxed{-\frac{1}{3}}$$

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* Asymptotes (of $f(x)$)

- Vertical Asymptotes (VA): ($x = \#$) Simplify $f(x)$ if possible, then set the denominator equal to 0 and solve for $x$.
- Horizontal Asymptote (HA): ($y = \#$) Find $\lim_{x \to \infty} f(x)$ and $\lim_{x \to -\infty} f(x)$. If one (or both) are finite (meaning $\neq \infty$ or $-\infty$), that is the HA.
- Slant Asymptote (SA): ($y = mx + b$) Only occur when the highest power on $x$ in the numerator is exactly one more than the highest power on $x$ in the denominator. Find by polynomial division.

**Ex. 6** Find the asymptotes of $f(x) = \frac{2x^2}{x^2-1}$.

VA: $f(x)$ can't be simplified further.

$$x^2 - 1 = 0$$

$$\underbrace{(x-1)}_{=0}\underbrace{(x+1)}_{=0} = 0$$

$$\boxed{x = 1, x = -1}$$

HA: $\lim_{x \to \infty} \frac{2x^2}{x^2-1} = \lim_{x \to \infty} \frac{2x^2}{x^2} = \lim_{x \to \infty} 2 = 2$

$\lim_{x \to -\infty} \frac{2x^2}{x^2-1} = \lim_{x \to -\infty} \frac{2x^2}{x^2} = \lim_{x \to -\infty} 2 = 2$

$$\boxed{y = 2}$$

SA: Both numerator and denominator have $x^2$ as the highest power, so there is no SA. $\boxed{\text{NONE}}$

* In general, a function will not have both a horizontal and slant asymptote.

**Ex.7** Find the asymptotes of $f(x) = \frac{x^2+7x-1}{x-2}$.

VA: Can't simplify $f(x)$.

$x - 2 = 0$

$\boxed{x = 2}$

HA:
$$\lim_{x\to\infty} \frac{x^2+7x-1}{x-2} = \lim_{x\to\infty} \frac{x^2}{x} = \lim_{x\to\infty} x = \infty$$
$$\lim_{x\to-\infty} \frac{x^2+7x-1}{x-2} = \lim_{x\to-\infty} \frac{x^2}{x} = \lim_{x\to-\infty} x = -\infty$$
$\boxed{\text{NONE}}$

SA: $x^2$ in numerator and $x$ in denominator, so $f(x)$ has a SA.
Use polynomial division.

$$\begin{array}{r l}
 & x + 9 \\
x-2 \,\big) & x^2 + 7x - 1 \\
 & -(x^2 - 2x) \downarrow \\
 & \quad 9x - 1 \\
 & -(9x - 18) \\
 & \qquad 17
\end{array}$$

Subtract both terms.

(x goes into $x^2$ $\frac{x^2}{x} = x$ times)

$x(x-2) = x^2 - 2x$

(x goes into $9x$ $\frac{9x}{x} = 9$ times)

x does not go into 17, so we're done.

We have $\frac{x^2+7x-1}{x-2} = \underbrace{x+9}_{SA} + \underbrace{\frac{17}{x-2}}_{\text{remainder}}$, so $\boxed{y = x+9}$

**Ex.8** Find the asymptotes of $f(x) = \frac{1+4x^2+2x^3}{x^2+1}$.

VA: Can't simplify $f(x)$.
$x^2+1 = 0$ never happens, so $\boxed{\text{NONE}}$

HA:
$$\lim_{x\to\infty} \frac{1+4x^2+2x^3}{x^2+1} = \lim_{x\to\infty} \frac{2x^3}{x^2} = \lim_{x\to\infty} 2x = \infty$$
$$\lim_{x\to-\infty} \frac{1+4x^2+2x^3}{x^2+1} = \lim_{x\to-\infty} \frac{2x^3}{x^2} = \lim_{x\to-\infty} 2x = -\infty$$
$\boxed{\text{NONE}}$

SA: $x^3$ in numerator and $x^2$ in denominator, so $f(x)$ has a SA. We should rewrite numerator from highest to lowest power before doing polynomial division.

$$\begin{array}{r l}
 & 2x + 4 + \frac{-2x-3}{x^2+1} \\
x^2+1 \,\big) & 2x^3 + 4x^2 + 1 \\
 & -(2x^3 \quad \downarrow \quad +2x) \downarrow \\
 & \qquad 4x^2 - 2x + 1 \\
 & \quad -(4x^2 \quad \downarrow \quad +4) \\
 & \qquad\quad -2x - 3
\end{array}$$

$\boxed{y = 2x + 4}$

Remember long division:

$$\begin{array}{r l}
 & 159 \\
2 \,\big) & 319 \\
 & \underline{2}\downarrow \\
 & 11 \\
 & \underline{-10}\downarrow \\
 & \quad 19 \\
 & \underline{-18} \\
 & \quad\ 1
\end{array}$$

$\frac{319}{2} = 159 + \frac{1}{2}$
$= 159.5$