Lesson 23: Curve Sketching

$\ast$ To sketch a curve $f(x)$, find the following.

0. Domain – don't include any x-values in the intervals if they're not in the domain
1. x-intercepts (points)

   (when $\boldsymbol{y = 0}$)
2. y-intercepts (points)

   (when $\boldsymbol{x = 0}$)
3. increasing and decreasing intervals

   (use $\boldsymbol{f'(x)}$)
4. concave up and concave down intervals

   (use $\boldsymbol{f''(x)}$)
5. inflection points

   (use $\boldsymbol{f''(x)}$)
6. horizontal asymptotes (lines)

   (find $y = \lim\limits_{x \to \infty} f(x)$ and $y = \lim\limits_{x \to -\infty} f(x)$)
7. vertical asymptotes (lines)

   (simplify $f(x)$, set the denominator equal to zero, and solve for $x = \#$)
8. slant asymptote (lines)

   (use polynomial division)

Note: It's usually best to plot the x- and y-intercepts and all the asymptotes first. Then use the increasing/decreasing and concave up/down intervals to figure out how to draw the curve (or on exams, how to pick which graph corresponds to the given function).

Ex. 1 Sketch the graph for $f(x) = \frac{x^2-2x+4}{x-2}$.

0. Domain: $(-\infty, 2) \cup (2, \infty)$
1. x-intercept(s): NONE
2. y-intercept(s): $(0, -2)$
3. Increasing Intervals: $(-\infty, 0) \cup (4, \infty)$

   Decreasing Intervals: $(0, 2) \cup (2, 4)$
4. Concave Up Intervals: $(2, \infty)$

   Concave Down Intervals: $(-\infty, 2)$
5. Inflection Point(s): NONE
6. Horizontal Asymptote: NONE
7. Vertical Asymptote: $x = 2$
8. Slant Asymptote: $y = x$

0. Set denominator equal to 0, and solve for x.

$$x - 2 = 0$$
$$x = 2$$

So $x = 2$ is not in the domain.

1. Set $y = 0$ (or $f(x) = 0$) and solve for the x-intercept x-coordinate.

$$0 = \frac{x^2 - 2x + 4}{x - 2}$$ (Remember, a fraction can only be zero if the numerator = 0)

$$0 = x^2 - 2x + 4 \rightarrow$$ Use quadratic formula.

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(4)}}{2(1)}$$

$$= \frac{2 \pm \sqrt{-12}}{2} \rightarrow$$ negative under a square root, so the x-intercepts don't exist.

2. Set $x = 0$ and solve for the y-intercept y-coordinate.

$$f(0) = \frac{0^2 - 2(0) + 4}{0 - 2} = \frac{4}{-2} = -2$$

3. Find $f'(x)$. Solve $f'(x) = 0$. Make a number line, and label where $f'(x) = 0$ and any domain issues. Then find the signs of $f'(x)$.

Quotient Rule:

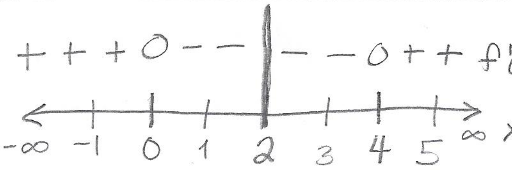
Top $= x^2 - 2x + 4$ Bottom $= x - 2$

Top' $= 2x - 2$ Bottom' $= 1$

$$f'(x) = \frac{T' \cdot B - T \cdot B'}{B^2} = \frac{(2x-2)(x-2) - (x^2 - 2x + 4)(1)}{(x-2)^2}$$

$$0 = \frac{2x^2 - 6x + 4 - x^2 + 2x - 4}{(x-2)^2}$$

$$0 = \frac{x^2 - 4x}{(x-2)^2} = \frac{x(x-4)}{(x-2)^2} \Rightarrow \begin{matrix} x = 0 \\ x = 4 \end{matrix}$$



$x = 2$ is not in the domain, so it will not be included in any interval.

4. Find $f''(x)$. Solve $f''(x) = 0$. Make a number line, and label where $f''(x) = 0$ and any domain issues. Then find the signs of $f''(x)$.

Quotient Rule:

Top $= x^2 - 4x$ Bottom $= (x-2)^2$

Top' $= 2x - 4$ Chain: Out $= x^2$ In $= x - 2$

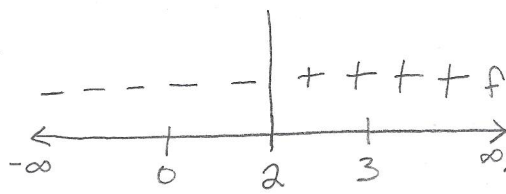
Out' $= 2x$ In' $= 1$

Bottom' $= 2(x-2)(1)$

$$f''(x) = \frac{T' \cdot B - T \cdot B'}{B^2} = \frac{(2x-4)(x-2)^2 - (x^2 - 4x)2(x-2)}{((x-2)^2)^2}$$

$$0 = \frac{\cancel{(x-2)}[(2x-4)(x-2) - 2(x^2 - 4x)]}{(x-2)^{\cancel{4}\,3}} = \frac{\cancel{2x^2} - 8x + 8 - \cancel{2x^2} + \cancel{8x}}{(x-2)^3}$$

$$0 = \frac{8}{(x-2)^3} \Rightarrow$$ Never equals 0.



5. Since f(x) changes concavity on either side of an x-value not in the domain, f(x) has no inflection points.

6. $\lim_{x\to\infty} \frac{x^2-2x+4}{x-2} = \lim_{x\to\infty} \frac{x^2}{x} = \lim_{x\to\infty} x = \infty$

$\lim_{x\to-\infty} \frac{x^2-2x+4}{x-2} = \lim_{x\to-\infty} \frac{x^2}{x} = \lim_{x\to-\infty} x = -\infty$

} NONE

7. $f(x) = \frac{x^2-2x+4}{x-2}$ can't be simplified,

so set denominator equal to 0, and solve for x.

$x - 2 = 0$

$x = 2$

8. Highest power in numerator = 2
" " denominator = 1 } exactly 1 more ⇒ f(x) has a SA

$$
\begin{array}{r}
x + \frac{4}{x-2} \\
x-2 \overline{\big)\, x^2 - 2x + 4} \\
-(x^2 - 2x) \downarrow \\
\hline
0 + 4
\end{array}
$$

Ex. 2 Sketch the graph for $f(x) = \frac{x^2+2}{x^2-1}$.

0. Domain: $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$
1. x-intercept(s): NONE
2. y-intercept(s): $(0, -2)$
3. Increasing Intervals: $(-\infty, -1) \cup (-1, 0)$

   Decreasing Intervals: $(0, 1) \cup (1, \infty)$
4. Concave Up Intervals: $(-\infty, -1) \cup (1, \infty)$

   Concave Down Intervals: $(-1, 1)$
5. Inflection Point(s): NONE
6. Horizontal Asymptote: $y = 1$
7. Vertical Asymptote: $x = -1$, $x = 1$
8. Slant Asymptote: NONE

0. $x^2 - 1 = 0$

   $(x-1)(x+1) = 0$

   $x = 1,\ x = -1 \rightarrow$ Not in domain.

1. $0 = \frac{x^2+2}{x^2-1}$

   $0 = x^2 + 2$

   $x^2 = -2 \longrightarrow$ Never happens.

2. $f(0) = \frac{0^2+2}{0^2-1} = \frac{2}{-1} = -2$

3. $f'(x) = \frac{2x(x^2-1) - (x^2+2)(2x)}{(x^2-1)^2}$

   $0 = \frac{2x^3 - 2x - 2x^3 - 4x}{(x^2-1)^2}$

   $0 = \frac{-6x}{(x^2-1)^2} \Rightarrow x = 0$

   + + + + | + + 0 − − | − − − − $f'(x)$

   $-\infty$ -2 -1 0 1 2 ∞ x

   Not in domain

4. $f''(x) = \frac{-6(x^2-1)^2 - (-6x)\,2(x^2-1)(2x)}{((x^2-1)^2)^2} = \frac{(x^2-1)[-6(x^2-1) + (6x)(2)(2x)]}{(x^2-1)^4}$

   $0 = \frac{-6x^2 + 6 + 24x^2}{(x^2-1)^3}$

   $0 = \frac{18x^2 + 6}{(x^2-1)^3} \Rightarrow$ Never $= 0$

   + + + | − − − − − | + + + + $f''(x)$

   $-\infty$ -2 -1 0 1 2 ∞ x

5. $f(x)$ changes concavity around domain issues, so no inflection points.

6. $\lim_{x \to \infty} \frac{x^2+2}{x^2-1} = \lim_{x \to \infty} \frac{x^2}{x^2} = \lim_{x \to \infty} 1 = 1 \Rightarrow y = 1$

7. Can't simplify $f(x)$.

   $0 = x^2 - 1$

   $0 = (x-1)(x+1)$

   $x = 1,\ x = -1$

8. Highest power in numerator = Highest power in denominator, so no SA.

Ex. 3 Sketch the graph for $f(x) = \frac{x+4}{x-2}$.

0. Domain: $(-\infty, 2) \cup (2, \infty)$
1. x-intercept(s): $(-4, 0)$
2. y-intercept(s): $(0, -2)$
3. Increasing Intervals: NONE

   Decreasing Intervals: $(-\infty, 2) \cup (2, \infty)$
4. Concave Up Intervals: $(2, \infty)$

   Concave Down Intervals: $(-\infty, 2)$
5. Inflection Point(s): NONE
6. Horizontal Asymptote: $y = 1$
7. Vertical Asymptote: $x = 2$
8. Slant Asymptote: NONE

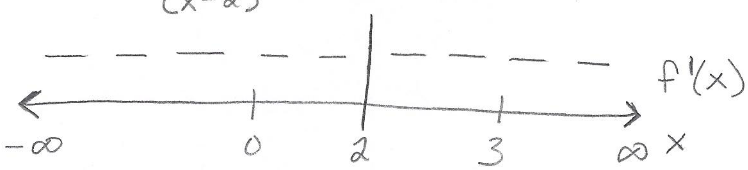
0. $x - 2 = 0$
   $x = 2$ is not in domain.

1. $0 = \frac{x+4}{x-2}$
   $0 = x + 4$
   $x = -4$

2. $f(0) = \frac{0+4}{0-2} = \frac{4}{-2} = -2$

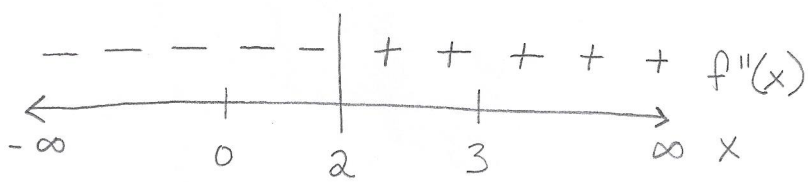
3. $f'(x) = \frac{(1)(x-2) - (x+4)(1)}{(x-2)^2} = \frac{x-2-x-4}{(x-2)^2}$

   $0 = \frac{-6}{(x-2)^2} \Rightarrow$ Never $= 0$



4. $f''(x) = -6(-2)(x-2)^{-3}(1) = \frac{12}{(x-2)^3}$

   $0 = \frac{12}{(x-2)^3} \Rightarrow$ Never $= 0$



5. $f(x)$ changes concavity around a domain issue, so none.

6. $\lim_{x \to \infty} \frac{x+4}{x-2} = \lim_{x \to \infty} \frac{x}{x} = \lim_{x \to \infty} 1 = 1 \Rightarrow y = 1$

7. Can't simplify $f(x)$, so $x - 2 = 0$
   $x = 2$

8. Highest power in numerator = Highest power in denominator, so no SA

Ex. 4 Sketch the graph for $f(x) = \frac{4}{x^2+2x+1}$.

0. Domain: $(-\infty, -1) \cup (-1, \infty)$
1. x-intercept(s): NONE
2. y-intercept(s): $(0, 4)$
3. Increasing Intervals: $(-\infty, -1)$

   Decreasing Intervals: $(-1, \infty)$
4. Concave Up Intervals: $(-\infty, -1) \cup (-1, \infty)$

   Concave Down Intervals: NONE
5. Inflection Point(s): NONE
6. Horizontal Asymptote: $y = 0$
7. Vertical Asymptote: $x = -1$
8. Slant Asymptote: NONE

<u>Note</u>: $f(x) = \frac{4}{x^2+2x+1} = \frac{4}{(x+1)^2} = 4(x+1)^{-2}$

0. $(x+1)^2 = 0$

   $x + 1 = 0$

   $x = -1$ not in domain.

1. $0 = \frac{4}{x^2+2x+1}$

   $0 = 4$ never happens.

2. $f(0) = \frac{4}{0^2+2(0)+1} = \frac{4}{1} = 4$

3. $f'(x) = 4(-2)(x+1)^{-3}(1) = \frac{-8}{(x+1)^3}$

   $0 = \frac{-8}{(x+1)^3} \Rightarrow$ never $= 0$

   $+ + + + \mid - - - -$ $\quad f'(x)$

   $-\infty \quad -2 \quad -1 \quad 0 \quad \infty$ $\quad x$

4. $f''(x) = -8(-3)(x+1)^{-4}(1) = \frac{24}{(x+1)^4}$

   $0 = \frac{24}{(x+1)^4} \Rightarrow$ never $= 0$

   $+ + + + \mid + + + +$ $\quad f''(x)$

   $-\infty \quad -2 \quad -1 \quad 0 \quad \infty$ $\quad x$

5. $f(x)$ never changes concavity.

6. $\lim_{x \to \infty} \frac{4}{x^2+2x+1} = 0 \Rightarrow y = 0$

7. Can't simplify $f(x)$.

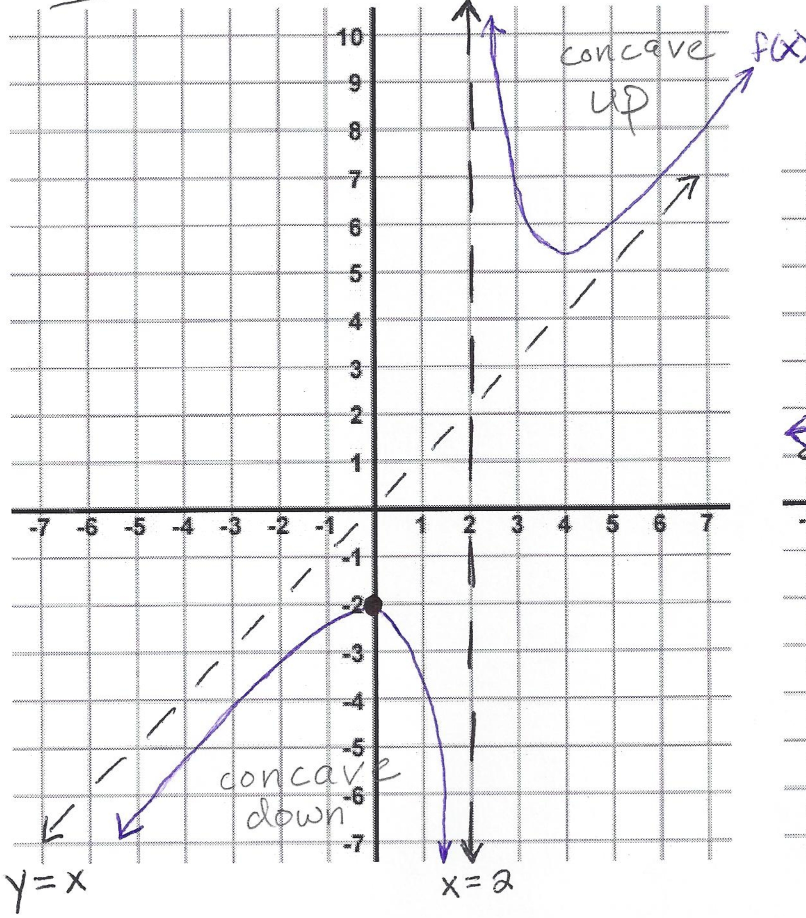
   $(x+1)^2 = 0$

   $x + 1 = 0$

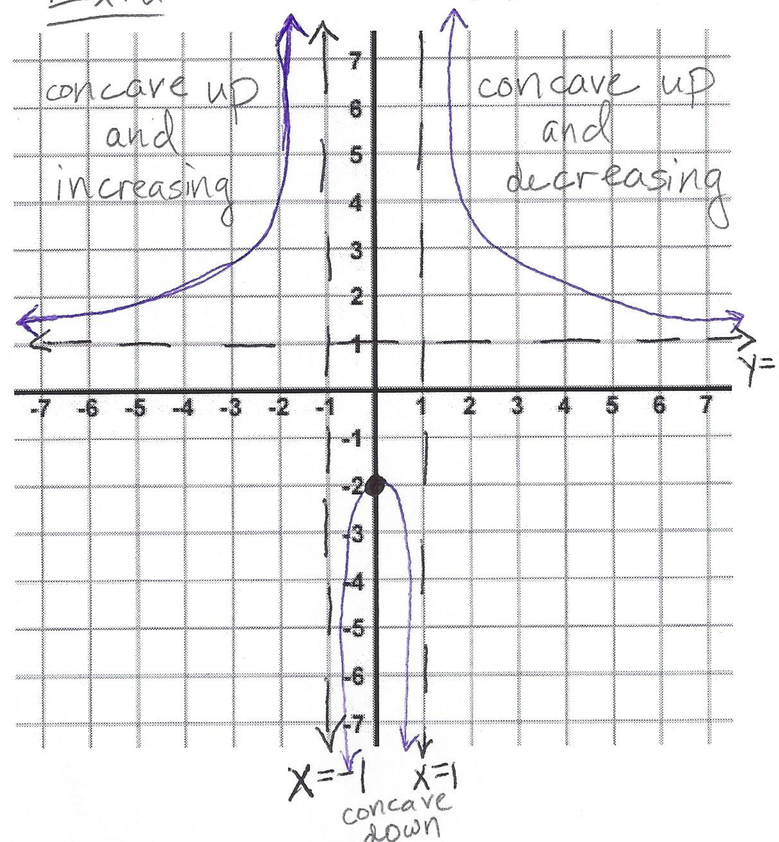
   $x = -1$

8. Highest power in numerator < Highest power in denominator, so no SA.

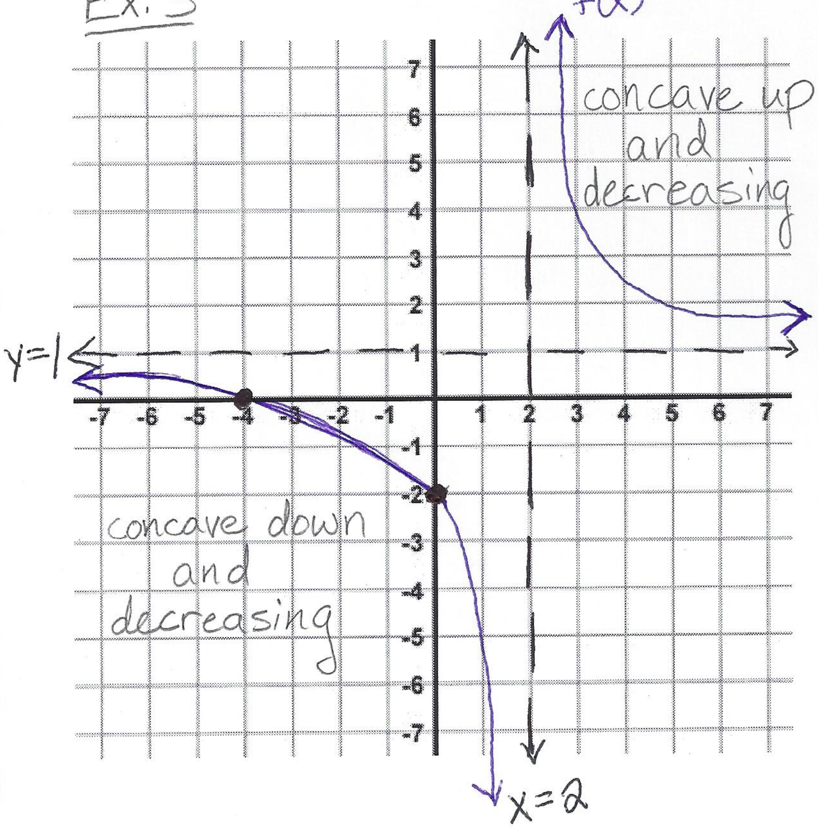
Ex. 1



Ex. 2



Ex. 3



Ex. 4