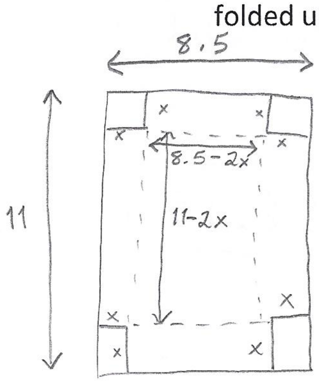
Lesson 24: Optimization (Part 1)

Steps: Draw a picture!

1. Identify the quantity to be optimized and the quantity that we can control/change (our variable).
2. Identify any constraints.
3. Use the constraints to write an *objective function* for the quantity we want to optimize in terms of a single variable (probably x).
4. Take the derivative of the objective function with respect to the variable.
5. Set the derivative equal to zero and solve for the variable. → usually easier!
   (Note: we *should* use the first or second derivative test to see whether the value we got is the minimum or the maximum, but usually, there will only be one solution that makes sense. If you have time on an exam, you should double check this.)
6. Compute the desired quantity. (This may **not** be the value you found. You may have to use the value you found to compute the desired quantity.)

Note: Optimization is different from related rates because we aren't worried about constants and variables with respect to *time*, instead we are concerned with what values we can control and what values/relations are fixed.

Ex. 1 A piece of cardstock is 8.5 inches by 11 inches. A square is to be cut from each corner and the sides folded up to make an open-top box. What is the maximum possible volume of the box?



① Maximize volume.
Control size of the square x.
② Constraint is size of the cardstock.

③ Volume = $(8.5-2x)(11-2x)x$

We've already included the constraints by using the picture.

$V = (8.5-2x)(11-2x)x$

$V = (93.5 - 22x - 17x + 4x^2)x$

$\boxed{V = 93.5x - 39x^2 + 4x^3}$ objective function

④ $\frac{dV}{dx} = V' = 93.5 - 78x + 12x^2$

⑤ $0 = 12x^2 - 78x + 93.5$

$x = \frac{-(-78) \pm \sqrt{(-78)^2 - 4(12)(93.5)}}{2(12)} = \frac{78 \pm \sqrt{1596}}{24}$

$\underbrace{\frac{78+\sqrt{1596}}{24} \approx 4.91}$ and $\frac{78-\sqrt{1596}}{24} \approx 1.59$

The paper is only 8.5 inches, so here, we would have no side left.
This means 1.59 is the answer that makes sense.

Double check with:

1st DT:

max: $\frown$ at 1.59; min: $\smile$ at 4.915

$$\begin{array}{ccccccccccc} & + + + & + + + & 0 & - - - & - - - & 0 & + + + & V' \\ -\infty & 0 & & 1.59 & 2 & & 4.91 & 5 & x \end{array}$$

$$V'\big|_{x=0} = 93.5 - 78(0) + 12(0)^2 = 93.5$$

$$V'\big|_{x=2} = 93.5 - 78(2) + 12(2)^2 = -14.5$$

$$V'\big|_{x=5} = 93.5 - 78(5) + 12(5)^2 = 3.5$$

2nd DT: $V'' = -78 + 24x$

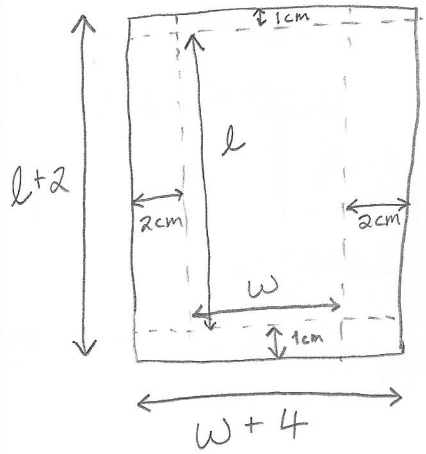
$$V''\big|_{x=1.59} = -78 + 24(1.59) = -39.84 \quad \frown \text{ max}$$

$$V''\big|_{x=4.91} = -78 + 24(4.91) = 39.84 \quad \smile \text{ min}$$

Both tests tell us that the max does indeed occur at $x = 1.59$ in.

6. $V \approx 93.5(1.59) - 39(1.59)^2 + 4(1.59)^3 \approx \boxed{66.15 \text{ in}^3}$

Ex. 2 Whitley and Jaleesa are designing a poster with area 400 cm^2 to contain a printing area having margins of 1 cm at the top and bottom and 2 cm on the side. Find the largest possible printing area.



① Maximize printing area $A = lw$.
Control poster dimensions l and w.

② Constraint is poster area $= 400 \text{ cm}^2$

$(l+2)(w+4) = 400$

③ Constraint: $400 = (l+2)(w+4)$ ← Solve for l or w.

$\frac{400}{w+4} = l + 2$

$\frac{400}{w+4} - 2 = l$ ← Plug into $A = lw$.

$$A = \left(\frac{400}{w+4} - 2\right)w$$

$$A = \frac{400w}{w+4} - 2w$$

④ $$\frac{dA}{dw} = A' = \frac{400(w+4) - 1(400w)}{(w+4)^2} - 2$$

$$= \frac{400w + 1600 - 400w}{(w+4)^2} - 2$$

$$= \frac{1600}{(w+4)^2} - 2$$

⑤ $$0 = \frac{1600}{(w+4)^2} - 2$$

$$2 = \frac{1600}{(w+4)^2}$$

$2(w+4)^2 = 1600$

$(w+4)^2 = 800$

$w + 4 = \pm\sqrt{800}$

$w = \pm\sqrt{800} - 4 \approx 24.28$ or $\cancel{-32.28}$

doesn't make sense to have negative width

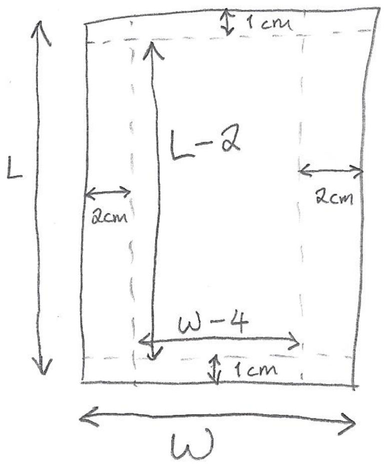
⑥ $$A = \frac{400w}{w+4} - 2w$$

$$A \approx \frac{400(24.28)}{24.28 + 4} - 2(24.28) \approx \boxed{294.86 \text{ cm}^2}$$

Note: Can check we have max with 1st DT or 2nd DT.
I'm going to use 2nd DT.

$A'' = 1600 \cdot (-2) \cdot \frac{1}{(w+4)^3} \cdot 1 = \frac{-3200}{(w+4)^3}$ $\quad A''\big|_{w=24.28} = -0.1414 < 0$ $\cap$ max

We could choose l and w to represent different quantities in our picture.



① Maximize printing area $A = (L-2)(W-4)$.
Control poster dimensions L and W.

② Constraint is poster area $400 = LW$.

③ Maximize: $A = (L-2)(W-4)$
Constraint: $400 = LW$ ← Solve for L or W.

$\frac{400}{W} = L$ ← Plug into $A=(L-2)(W-4)$

$$A = \left(\frac{400}{W} - 2\right)(W-4)$$
$$A = \frac{400}{\cancel{W}} \cdot \cancel{W} - \frac{1600}{W} - 2W + 8$$
$$A = -\frac{1600}{W} - 2W + 408$$

④ $$\frac{dA}{dW} = A' = \frac{1600}{W^2} - 2$$

⑤ $$0 = \frac{1600}{W^2} - 2$$
$$2 = \frac{1600}{W^2}$$
$$W^2 = 800$$
$$W = \pm\sqrt{800} \approx \pm 28.28$$ → Negative width does not make sense.

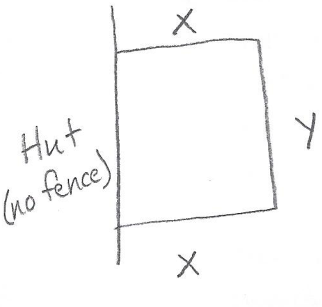
⑥ $$A = -\frac{1600}{\sqrt{800}} - 2\sqrt{800} + 408$$
$$\boxed{\approx 294.86 \text{ cm}^2}$$

Note: Again, we can double check that this is the maximum with 1st DT or 2nd DT.

$$A'' = -2 \cdot \frac{1600}{W^3} = -\frac{3200}{W^3}$$

$$A''\Big|_{W=\sqrt{800}} = -\frac{3200}{(\sqrt{800})^3} \approx -0.1414 < 0 \quad \overset{\text{max}}{\cap}$$

Ex. 3 Hagrid has $10T$ feet of fence to make a rectangular pumpkin patch alongside the wall of his hut, where T is a positive constant. The wall of the hut bounds one side of the pumpkin patch. What is the largest possible area for the pumpkin patch?



① Maximize area
Control dimensions of pumpkin patch.

② Constraint is amount of fence $= 10T$

③ $A = xy$

$10T = 2x + y$ solve for x or y.

$y = 10T - 2x$

plug into area equation

Treat T as a constant when taking derivative

$A = x(10T - 2x)$

$A = 10Tx - 2x^2$

④ $\frac{dA}{dx} = A' = 10T - 4x$

⑤ $0 = 10T - 4x$

$4x = 10T$

$x = \frac{5T}{2}$

⑥ $A = 10T\left(\frac{5T}{2}\right) - 2\left(\frac{5T}{2}\right)^2$

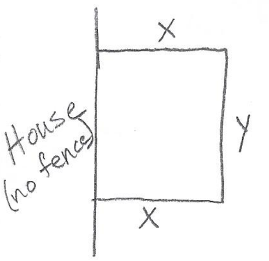
$= 25T^2 - 2 \cdot \frac{25T^2}{4}$

$= 25T^2 - \frac{25T^2}{2} = \boxed{\frac{25T^2}{2}}$

Note: Use 1st DT or 2nd DT to check for max.

2nd DT: $A'' = -4 < 0$ ∩ max

Ex. 4 Stu and Didi want to fence a rectangular play area alongside the wall of their house. The wall of their house bounds one side of the play area. If they want the play area to be exactly 1000 ft^2, what is the least amount of fencing needed to make this? (Answer: 89.44 ft)



① Minimize fencing.
Control dimensions

② Constraint is area $= 1000$ ft^2

③ $A = xy$

$1000 = xy$

$\frac{1000}{x} = y$

$P = 2x + y$

$P = 2x + \frac{1000}{x}$

④ $P' = 2 - \frac{1000}{x^2}$

⑤ $0 = 2 - \frac{1000}{x^2}$

$\frac{1000}{x^2} = 2$

$1000 = 2x^2$

$500 = x^2$

$\pm\sqrt{500} = x$

Negative answer doesn't make sense.

⑥ $P = 2(\sqrt{500}) + \frac{1000}{\sqrt{500}} \approx \boxed{89.44 \text{ ft}}$

Note: Use 1st or 2nd DT to check that this is where the min occurs.

2nd DT: $P'' = -(-2)\frac{1000}{x^3} = \frac{2000}{x^3}$

$P''\big|_{x=\sqrt{500}} = \frac{2000}{500^{3/2}} = 0.179 > 0$

∪ min