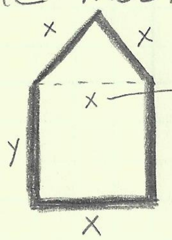
Lesson 25: Optimization (Part 2)

Ex. 1 Mr. Woolsey is adding a window to the Atlantis conference room. He wants to construct it by adjoining an equilateral triangular window to the top of a rectangular window. If the perimeter of the window is 32 ft, find the dimensions (x and y) that will allow the window to admit the most light.



① Maximize the area of the window.

Total Area = Area □ + Area △

$$A = xy + \frac{\sqrt{3}}{4}x^2$$

Note: Need to know the formula for the area of a rectangle, but would be given the formula for an equilateral triangle.

② Constraint is the perimeter of the window is 32.

$$32 = 3x + 2y$$

③ Solve the constraint for x or y.

Note: We should solve for y because we will plug this equation into the equation for A in ①. There is only one y in that equation, but it has two x's and one is squared, so it will be easier to plug in an expression for y than for x.

$$32 = 3x + 2y$$

$$\frac{32 - 3x}{2} = \frac{2y}{2}$$

$$\frac{32}{2} - \frac{3x}{2} = y$$

$$16 - \frac{3}{2}x = y$$

Plug $y = 16 - \frac{3}{2}x$ into $A = xy + \frac{\sqrt{3}}{4}x^2$ (from ①).

$$A = x\left(16 - \frac{3}{2}x\right) + \frac{\sqrt{3}}{4}x^2$$

$$A = 16x - \frac{3}{2}x^2 + \frac{\sqrt{3}}{4}x^2$$

④ Take the derivative of A.

$$A' = 16 - \frac{3}{2}(2x) + \frac{\sqrt{3}}{4}(2x)$$

$$A' = 16 - 3x + \frac{\sqrt{3}}{2}x$$

5 Set $A' = 0$ and solve for x.

$$16 - 3x + \frac{\sqrt{3}}{2}x = 0$$

$$16 = 3x - \frac{\sqrt{3}}{2}x$$

$$16 = \left(3 - \frac{\sqrt{3}}{2}\right)x$$

$$16 = \left(\frac{6}{2} - \frac{\sqrt{3}}{2}\right)x$$

$$16 = \left(\frac{6 - \sqrt{3}}{2}\right)x$$

$$\frac{16}{\left(\frac{6-\sqrt{3}}{2}\right)} = x$$

$$\frac{16}{1} \cdot \frac{2}{(6-\sqrt{3})} = x$$

$$\frac{32}{6-\sqrt{3}} = x$$

6 Find the dimensions x and y.

$$\boxed{x = \frac{32}{6-\sqrt{3}}}$$

From step 3, $y = 16 - \frac{3}{2}x$

$$y = 16 - \frac{3}{\cancel{2}_{1}}\left(\frac{\overset{16}{\cancel{32}}}{6-\sqrt{3}}\right)$$

$$y = 16 - \frac{3(16)}{6-\sqrt{3}}$$

Find a common denominator.

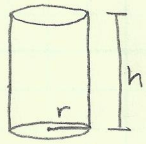
$$y = \frac{16}{1} \cdot \frac{6-\sqrt{3}}{6-\sqrt{3}} - \frac{48}{6-\sqrt{3}}$$

$$y = \frac{16(6-\sqrt{3}) - 48}{6-\sqrt{3}}$$

$$y = \frac{96 - 16\sqrt{3} - 48}{6-\sqrt{3}}$$

$$\boxed{y = \frac{48 - 16\sqrt{3}}{6-\sqrt{3}}}$$

Ex. 2 A box has a circular base. If the sum of the height of the box and the perimeter of the circular base is 27π in, what is the maximum possible volume?



① Maximize the volume of the box.

$$\text{Volume} = \underset{(\text{circle})}{\text{Area of Base}} \times \text{Height}$$

$$V = \pi r^2 h$$

② Constraint: Height of the box + Perimeter of the base (circle) $= 27\pi$

$$h + 2\pi r = 27\pi$$

③ Solve the constraint for h or r.

Note: We should solve for h because it is not squared in the volume equation.

$$h + 2\pi r = 27\pi$$
$$h = 27\pi - 2\pi r$$

Plug $h = 27\pi - 2\pi r$ into the equation for V in ①.

$$V = \pi r^2 h$$
$$V = \pi r^2(27\pi - 2\pi r)$$
$$V = 27\pi^2 r^2 - 2\pi^2 r^3$$

④ Take the derivative of V.

$$\frac{dV}{dr} = 27\pi^2(2r) - 2\pi^2(3r^2)$$
$$\frac{dV}{dr} = 54\pi^2 r - 6\pi^2 r^2$$

⑤ Set $V' = 0$ and solve for r.

$$54\pi^2 r - 6\pi^2 r^2 = 0$$
$$6\pi^2 r(9 - r) = 0, \text{ so}$$

$6\pi^2 = 0$ never happens

$r = 0$ doesn't make sense

OR

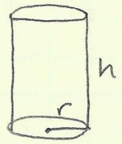
$9 - r = 0 \Rightarrow r = 9$

⑥ Need to find the maximum volume.

Plug $r = 9$ into the equation for V from step ③.

$$V = 27\pi^2(9)^2 - 2\pi^2(9)^3$$
$$V = 2187\pi^2 - 1458\pi^2$$
$$\boxed{V = 729\pi^2}$$

Ex. 3 Sam and Jack have a cylindrical Jello mold with a surface area of 25. What is the maximum volume that it can have?



① Maximize volume.

$$V = \pi r^2 h \quad \text{(Equation would be } \underline{\text{given}} \text{ on an exam.)}$$

② Constraint: Surface Area $= 25$

$$2\pi r^2 + 2\pi r h = 25 \quad \text{(Equation } \underline{\text{given}} \text{ on exam.)}$$

③ Solve the constraint for r or h.
Note: Since the constraint has an r in two terms, h will be easier to solve for than r.

$$2\pi r^2 + 2\pi r h = 25$$

$$\frac{2\pi r h}{2\pi r} = \frac{25 - 2\pi r^2}{2\pi r}$$

$$h = \frac{25}{2\pi r} - \frac{2\pi r^2}{\cancel{2\pi r}}$$

$$h = \frac{25}{2\pi r} - r$$

Plug $h = \frac{25}{2\pi r} - r$ into the equation for V in ①.

$$V = \pi r^2 h$$

$$V = \pi r^2 \left(\frac{25}{2\pi r} - r\right)$$

$$V = \frac{25\cancel{\pi} r^{\cancel{2}}}{2\cancel{\pi}\cancel{r}} - \pi r^3$$

$$V = \frac{25}{2} r - \pi r^3$$

④ Take the derivative of V.

$$V' = \frac{25}{2} - \pi(3r^2)$$

⑤ Set $V' = 0$ and solve for r.

$$0 = \frac{25}{2} - 3\pi r^2$$

$$\frac{3\pi r^2}{3\pi} = \frac{\frac{25}{2}}{3\pi}$$

$$r^2 = \frac{25}{2} \cdot \frac{1}{3\pi} = \frac{25}{6\pi}$$

$$r = \pm\sqrt{\frac{25}{6\pi}} \longrightarrow \text{Only } r = \sqrt{\frac{25}{6\pi}} \text{ makes sense.}$$

⑥ Find the maximum volume. Plug $r = \sqrt{\frac{25}{6\pi}}$ into V from ③.

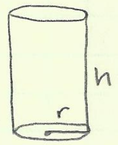
$$V = \frac{25}{2}\left(\sqrt{\frac{25}{6\pi}}\right) - \pi\left(\sqrt{\frac{25}{6\pi}}\right)^3$$

$$\boxed{V \approx 9.597}$$

Ex. 4 The SGC needs to make a circular can that can hold precisely 4.1 L of liquid naquadah. If the entire can is to be made out of trinium, find the dimensions (in cm) of the can that will minimize the cost.

Note: The given volume is 4.1 L, but we need the answer in cm, so we should convert the given volume from liters to cm^3.

$$4.1\ L \times \frac{1000\ mL}{1\ L} \times \frac{1\ cm^3}{1\ mL} = 4100\ cm^3$$



1 Minimize cost of material, so minimize surface area.

$$S = 2\pi r^2 + 2\pi r h \quad \text{(Equation given on exam.)}$$

2 Constraint : Volume $= 4100\ cm^3$

$$\pi r^2 h = 4100 \quad \text{(Equation given on exam.)}$$

3 Solve the constraint for r or h.
Note: Easier to solve for h.

$$\pi r^2 h = 4100$$
$$h = \frac{4100}{\pi r^2}$$

Plug $h = \frac{4100}{\pi r^2}$ into equation for S in 1.

$$S = 2\pi r^2 + 2\cancel{\pi}\cancel{r}\left(\frac{4100}{\cancel{\pi} r^{\cancel{2}}}\right)$$
$$S = 2\pi r^2 + \frac{8200}{r} \quad (= 2\pi r^2 + 8200 r^{-1})$$

4 Take derivative of S.

$$S' = 2\pi(2r) + 8200(-r^{-2})$$

5 Set $S' = 0$ and solve for r.

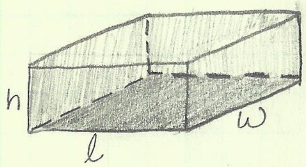
$$4\pi r - \frac{8200}{r^2} = 0$$
$$4\pi r = \frac{8200}{r^2}$$
$$4\pi r^3 = 8200$$
$$r^3 = \frac{8200}{4\pi} = \frac{2050}{\pi}$$
$$r = \sqrt[3]{\frac{2050}{\pi}}$$

6 Find dimensions r and h.

$$r = \sqrt[3]{\frac{2050}{\pi}}$$
$$\boxed{r \approx 8.67\ cm}$$

$$h = \frac{4100}{\pi r^2} \quad \text{(from 3)}$$
$$h = \frac{4100}{\pi\left(\frac{2050}{\pi}\right)^{2/3}}$$
$$\boxed{h \approx 17.35\ cm}$$

Ex.5 A box with a square base and no top is to be built with a volume of 4 in^3. Find the dimensions (length, width, and height) of the box that require the least amount of material. How much material is required at the minimum?



① Minimize surface area.

$$S = \underbrace{lw}_{\text{bottom}} + \underbrace{2lh}_{\text{front and back}} + \underbrace{2wh}_{\text{left and right}}$$

(Note: Think in terms of square base first, so $S = x^2 + 4xh$.)

② Constraint(s): Volume $= 4$ $\quad$ Square base: $l = w$

$lwh = 4$

③ First we can use $l = w$ to get rid of one of them.

Constraint: $w^2 h = 4$ $\quad$ Minimize: $S = w^2 + 2wh + 2wh$

$S = w^2 + 4wh$

Solve constraint $w^2 h = 4$ for w or h.

Note: Easier to solve for h.

$$w^2 h = 4$$
$$h = \frac{4}{w^2}$$

Plug $h = \frac{4}{w^2}$ into equation for $S = w^2 + 4wh$.

$$S = w^2 + 4w\left(\frac{4}{w^2}\right)$$
$$S = w^2 + \frac{16}{w} \quad (= w^2 + 16w^{-1})$$

④ Take derivative of S.

$$S' = 2w + 16(-w^{-2})$$

⑤ Set $S' = 0$ and solve for w.

$$2w - \frac{16}{w^2} = 0$$
$$2w = \frac{16}{w^2}$$
$$2w^3 = 16$$
$$w^3 = 8$$
$$w = 2$$

⑥ Find dimensions: $l = w = 2$

$$h = \frac{4}{w^2} = \frac{4}{2^2} = 1$$

$$\boxed{\begin{array}{c} l = w = 2 \text{ in} \\ h = 1 \text{ in} \end{array}}$$

Find surface area: (Plug $w = 2$ into equation for S in ③.)

$$S = (2)^2 + \frac{16}{2}$$
$$S = 4 + 8$$
$$\boxed{S = 12 \text{ in}^2}$$