Mr. Woolsey is adding a window to the Atlantis conference room. He wants to construct it by adjoining an equilateral triangular window to the top of a rectangular window. If the perimeter of the window is 32 ft, and the dimensions \((x, y)\) that will allow the window to admit the most light.

1. **Maximize the area of the window.**
   
   \[
   \text{Total Area} = \text{Area of rectangle} + \text{Area of triangle}
   \]
   
   \[
   A = xy + \frac{\sqrt{3}}{4} x^2
   \]
   
   **Note:** Need to know the formula for the area of a rectangle, but would be given the formula for an equilateral triangle.

2. **Constraint is the perimeter of the window is 32.**
   
   \[
   32 = 3x + 2y
   \]

3. **Solve the constraint for \(x\) or \(y\).**
   
   **Note:** We should solve for \(y\) because we will plug this equation into the equation for \(A\) in 1. There is only one \(y\) in that equation, but it has two \(x\)'s and one is squared, so it will be easier to plug in an expression for \(y\) than for \(x\).

   \[
   \frac{32 - 3x}{2} = \frac{2y}{2}
   \]

   \[
   \frac{32 - 3x}{2} = y
   \]

   \[
   16 - \frac{3}{2}x = y
   \]

   Plug \(y = 16 - \frac{3}{2}x\) into \(A = xy + \frac{\sqrt{3}}{4} x^2\) (from 1).

   \[
   A = x(16 - \frac{3}{2}x) + \frac{\sqrt{3}}{4} x^2
   \]

   \[
   A = 16x - \frac{3}{2}x^2 + \frac{\sqrt{3}}{4} x^2
   \]

4. **Take the derivative of \(A\).**

   \[
   A' = 16 - \frac{3}{2}(2x) + \frac{\sqrt{3}}{4}(2x)
   \]

   \[
   A' = 16 - 3x + \frac{\sqrt{3}}{2} x
   \]
5. Set $A' = 0$ and solve for $x$.

$$16 - 3x + \frac{13}{2}x = 0$$

$$16 = 3x - \frac{13}{2}x$$

$$16 = (3 - \frac{13}{2})x$$

$$16 = (\frac{6}{2} - \frac{13}{2})x$$

$$16 = \frac{32}{6-13}x$$

$$\frac{16}{(6-13)} = x$$

$$\frac{16}{1} \cdot \frac{2}{(6-13)} = x$$

$$\frac{32}{6-13} = x$$

6. Find the dimensions $x$ and $y$.

$$x = \frac{32}{6-13}$$

From step 3, 

$$y = 16 - \frac{3}{2}x$$

$$y = 16 - \frac{3}{2} \left( \frac{32}{6-13} \right)$$

$$y = 16 - \frac{3 \cdot 32}{6-13}$$

$$y = \frac{16 \cdot 6-13}{6-13} - \frac{48}{6-13}$$

$$y = \frac{16(6-13) - 48}{6-13}$$

$$y = \frac{96 - 16 \cdot 13 - 48}{6-13}$$

$$y = \frac{48 - 16 \cdot 13}{6-13}$$

Find a common denominator.
Ex. 2 A box has a circular base. If the sum of the height of the box and the perimeter of the circular base is $27\pi$ in, what is the maximum possible volume?

1. Maximize the volume of the box.
   
   $\text{Volume} = \text{Area of Base} \times \text{Height (circle)}$
   
   $V = \pi r^2 \cdot h$

2. Constraint: Height of the box + Perimeter of the base (circle) = $27\pi$
   
   $h + 2\pi r = 27\pi$
   $h = 27\pi - 2\pi r$

3. Solve the constraint for $h$ or $r$.
   
   Note: We should solve for $h$ because it is not squared in the volume equation.
   
   $h + 2\pi r = 27\pi$
   $h = 27\pi - 2\pi r$

   Plug $h = 27\pi - 2\pi r$ into the equation for $V$ in 1.

   $V = \pi r^2 \cdot h$
   $V = \pi r^2 (27\pi - 2\pi r)$
   $V = 27\pi^2 r^2 - 2\pi^2 r^3$

4. Take the derivative of $V$.
   
   $\frac{dV}{dr} = 27\pi^2 (2r) - 2\pi^2 (3r^2)$
   $\frac{dV}{dr} = 54\pi^2 r - 6\pi^2 r^2$

5. Set $V' = 0$ and solve for $r$.
   
   $54\pi^2 r - 6\pi^2 r^2 = 0$
   
   $6\pi^2 r (9 - r) = 0$, so $6\pi^2 = 0$ never happens
   
   $r = 0$ doesn't make sense
   
   $9 - r = 0 \Rightarrow r = 9$

6. Need to find the maximum volume.
   
   Plug $r = 9$ into the equation for $V$ from step 3.

   $V = 27\pi^2 (9)^2 - 2\pi^2 (9)^3$
   $V = 2187\pi^2 - 1458\pi^2$
   $V = 729\pi^2$
Ex 3 Sam and Jack have a cylindrical Jello mold with a surface area of 25. What is the maximum volume that it can have?

1. Maximize volume.
   \[ V = \pi r^2 h \]  (Equation would be given on an exam.)

2. Constraint: Surface Area = 25
   \[ 2\pi r^2 + 2\pi rh = 25 \]  (Equation given on exam.)

3. Solve the constraint for \( r \) or \( h \).
   Note: Since the constraint has an \( r \) in two terms, \( h \) will be easier to solve for than \( r \).
   \[
   \begin{align*}
   2\pi r^2 + 2\pi rh &= 25 \\
   2\pi rh &= 25 - 2\pi r^2 \\
   h &= \frac{25}{2\pi} - \frac{2\pi r^2}{2\pi} \\
   h &= \frac{25}{2\pi} - r
   \end{align*}
   \]

   Plug \( h = \frac{25}{2\pi} - r \) into the equation for \( V \) in 1.
   \[
   \begin{align*}
   V &= \pi r^2 h \\
   V &= \pi r^2 \left( \frac{25}{2\pi} - r \right) \\
   V &= \frac{25\pi r^2}{2\pi} - \pi r^3 \\
   V &= \frac{25}{2} r - \pi r^3
   \end{align*}
   \]

4. Take the derivative of \( V \).
   \[ V' = \frac{25}{2} - \pi (3r^2) \]

5. Set \( V' = 0 \) and solve for \( r \).
   \[
   0 = \frac{25}{2} - 3\pi r^2 \\
   \frac{3\pi r^2}{3\pi} = \frac{25}{3\pi} \\
   r^2 = \frac{25}{3\pi} \\
   r = \pm \frac{\sqrt{25}}{\sqrt{3\pi}} \\
   \text{Only } r = \frac{5}{\sqrt{3\pi}} \text{ makes sense.}
   \]

6. Find the maximum volume. Plug \( r = \frac{5}{\sqrt{3\pi}} \) into \( V \) from 3.
   \[
   V = \frac{25}{2} \left( \frac{5}{\sqrt{3\pi}} \right) - \pi \left( \frac{5}{\sqrt{3\pi}} \right)^3
   \]
   \[ V \approx 9.597 \]
The SGC needs to make a circular can that can hold precisely 4.1 L of liquid naguadah. If the entire can is to be made out of tinium, find the dimensions (in cm) of the can that will minimize the cost.

**Note:** The given volume is 4.1 L, but we need the answer in cm, so we should convert the given volume from liters to cm³.

\[ 4.1 \text{ L} \times \frac{1000 \text{ mL}}{1 \text{ L}} \times \frac{1 \text{ cm}^3}{1 \text{ mL}} = 4100 \text{ cm}^3 \]

1. Minimize cost of material, so minimize surface area.
   \[ S = 2\pi r^2 + 2\pi rh \] (Equation given on exam)

2. Constraint: Volume = 4100 cm³
   \[ \pi r^2 h = 4100 \] (Equation given on exam)

3. Solve the constraint for r or h.
   Note: Easier to solve for h.
   \[ \pi r^2 h = 4100 \]
   \[ h = \frac{4100}{\pi r^2} \]

   Plug \( h = \frac{4100}{\pi r^2} \) into equation for S in 1.
   \[ S = 2\pi r^2 + 2\pi r \left( \frac{4100}{\pi r^2} \right) \]
   \[ S = 2\pi r^2 + \frac{8200}{r} \]
   \[ = 2\pi r^2 + 8200 r^{-1} \]

4. Take derivative of S.
   \[ S' = 2\pi (2r) + 8200 (-r^{-2}) \]

5. Set \( S' = 0 \) and solve for r.
   \[ 4\pi r - \frac{8200}{r^2} = 0 \]
   \[ 4\pi r = \frac{8200}{r^2} \]
   \[ 4\pi r^3 = 8200 \]
   \[ r^3 = \frac{8200}{4\pi} \]
   \[ r = \left( \frac{8200}{4\pi} \right)^{\frac{1}{3}} \]

6. Find dimensions r and h.
   \[ r = \left( \frac{8200}{4\pi} \right)^{\frac{1}{3}} \]
   \[ r \approx 8.67 \text{ cm} \]

   \[ h = \frac{4100}{\pi r^2} \text{ (from 3)} \]
   \[ h = \frac{4100}{\pi \left( \frac{8200}{4\pi} \right)^{\frac{2}{3}}} \]
   \[ h \approx 17.35 \text{ cm} \]
Ex. 5

A box with a square base and no top is to be built with a volume of 4 ft³. Find the dimensions (length, width, and height) of the box that require the least amount of material. How much material is required at the minimum?

1. Minimize surface area. (Note: Think in terms of square base first, so \( S = x^2 + 4xh \).

2. Constraint(s): Volume = 4
   \( l \times w \times h = 4 \)
   Square base: \( l = w \)

3. First we can use \( l = w \) to get rid of one of them.
   Constraint: \( w^2 h = 4 \)
   Minimize: \( S = w^2 + 2wh + 2wh \)
   \( S = w^2 + 4wh \)

Solve constraint \( w^2 h = 4 \) for \( w \) or \( h \).

Note: Easier to solve for \( h \).
   \( w^2 h = 4 \)
   \( h = \frac{4}{w^2} \)

Plug \( h = \frac{4}{w^2} \) into equation for \( S = w^2 + 4wh \).
   \( S = w^2 + 4w \left( \frac{4}{w^2} \right) \)
   \( S = w^2 + \frac{16}{w} \) (\( = w^2 + 16w^{-1} \))

4. Take derivative of \( S \).
   \( S' = 2w + 16 \left( -w^{-2} \right) \)

5. Set \( S' = 0 \) and solve for \( w \).
   \[ 2w - \frac{16}{w^2} = 0 \]
   \[ 2w = \frac{16}{w^2} \]
   \[ 2w^3 = 16 \]
   \[ w^3 = 8 \]
   \[ w = 2 \]

6. Find dimensions: \( l = w = 2 \) ft
   \( h = \frac{4}{w^2} = \frac{4}{2^2} = 1 \)

Find surface area: \( S = (2)^2 + \frac{16}{2} \)
(Plug \( w = 2 \) into equation for \( S \) in (3).
\[ S = 4 + 8 \]
\[ S = 12 \text{ ft}^2 \]