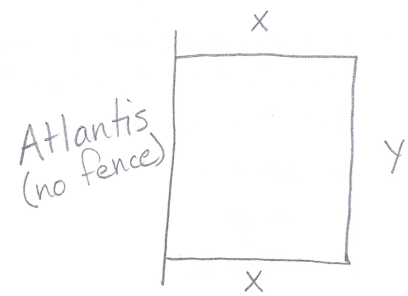
Lesson 26: Optimization (Part 3)



Ex 1 ① Minimize: Cost of fence

$C = 15(\text{Amount of fencing})$

$C = 15(2x + y)$

$C = 30x + 15y$

② Constraint: Area $= 2500 \text{ m}^2$

$xy = 2500$

③ Solve constraint for x or y: $y = \frac{2500}{x}$

Plug into $C = 30x + 15y$: $C = 30x + 15\left(\frac{2500}{x}\right)$

$C = 30x + \frac{37500}{x}$

④ Take derivative: $C' = 30 - \frac{37500}{x^2}$

⑤ Set equal to 0 and solve: $0 = 30 - \frac{37500}{x^2}$

$\frac{37500}{x^2} = 30$

$37500 = 30x^2$

$1250 = x^2$

$\pm\sqrt{1250} = x$

⑥ Find minimum cost: $C = 30x + \frac{37500}{x}$

$C = 30\sqrt{1250} + \frac{37500}{\sqrt{1250}}$

$\boxed{C \approx \$2{,}121.32}$

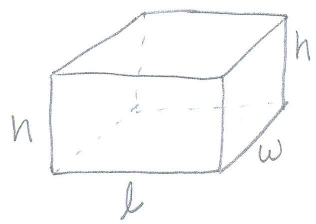
Use 1st or 2nd DT to check if min occurs at $x = \sqrt{1250}$:

2nd DT: $C'' = -(-2) \cdot \frac{37500}{x^3}$

$C'' = \frac{75000}{x^3}$

@ $x = \sqrt{1250}$: $C'' = \frac{75000}{(\sqrt{1250})^3} > 0$ ∪ min

Ex. 2



① Minimize Cost:

$$C = \underset{\substack{\uparrow \\ \text{cost of} \\ \text{trinium/cm}^2}}{20}(\text{area of sides}) + \underset{\substack{\uparrow \\ \text{cost of} \\ \text{ebony/cm}^2}}{5}(\text{area of top and bottom})$$

$$C = 20(\underset{\substack{\uparrow \\ \text{area of} \\ \text{front and} \\ \text{back}}}{2lh} + \underset{\substack{\uparrow \\ \text{area of} \\ \text{left and} \\ \text{right}}}{2wh}) + 5(\underset{\substack{\uparrow \\ \text{area of top} \\ \text{and bottom}}}{2lw})$$

$$C = 40lh + 40wh + 10lw$$

② Constraints: - Volume is 300 cm^3

$$lwh = 300$$

- length is 2 times width of base

$$l = 2w$$

③ We have 3 variables this time. Since both of the constraints have l and w, we should first get rid of h using $lwh = 300 \Rightarrow h = \frac{300}{lw}$.

Then $C = 40\cancel{l}\left(\frac{300}{\cancel{l}w}\right) + 40\cancel{w}\left(\frac{300}{l\cancel{w}}\right) + 10lw$

$$C = \frac{12000}{w} + \frac{12000}{l} + 10lw$$

Now use $l = 2w$ to get C entirely in terms of w.

$$C = \frac{12000}{w} + \frac{12000}{2w} + 10(2w)w$$

$$C = \frac{12000}{w} + \frac{6000}{w} + 20w^2$$

objective function $\rightarrow$ $C = \frac{18000}{w} + 20w^2$

④ Take derivative: $C' = -\frac{18000}{w^2} + 40w$

⑤ Set $=0$ and solve: $0 = -\frac{18000}{w^2} + 40w$

$$\frac{18000}{w^2} = 40w$$

$$18000 = 40w^3$$

$$450 = w^3$$

$$\sqrt[3]{450} = w$$

Use 1st or 2nd DT to check the min occurs here:

$$C'' = -(-2)\frac{18000}{w^3} + 40$$

$$= \frac{36000}{w^3} + 40$$

at $w = \sqrt[3]{450}$

$C'' > 0$ ∪ min

⑥ Find dimensions:

$w = \sqrt[3]{450}$ $\quad$ $l = 2 \cdot \sqrt[3]{450}$ $\quad$ $h = \frac{300}{lw} = \frac{300}{(7.66)(15.33)}$

$\boxed{w \approx 7.66 \quad l \approx 15.33 \quad h \approx 2.55}$

__Ex. 3__ First, maximize revenue:

(1) Revenue = price × units sold

$$R = pq$$

(2) Constraint: units sold = $q = 1800 - 50p$

(3) Plug q into R:

$$R = p(1800 - 50p)$$
$$R = 1800p - 50p^2$$

(4) Take derivative:

$$R' = 1800 - 100p$$

Check if max:
$R'' = -100 < 0$ max
by 2nd DT

(5) Set $= 0$ and solve:

$$0 = 1800 - 100p$$
$$100p = 1800$$
$$p = 18$$

(6) Should charge $\boxed{\$18/\text{ream}}$

Now, maximize profit:

(1) Profit = (price − cost) × units sold

$$P = (p-3)q$$

(2) Constraint: $q = 1800 - 50p)$

(3) Plug q into P:

$$P = (p-3)(1800-50p)$$
$$P = 1800p - 50p^2 - 5400 + 150p$$
$$P = 1950p - 50p^2 - 5400$$

(4) Take derivative:

$$P' = 1950 - 100p$$

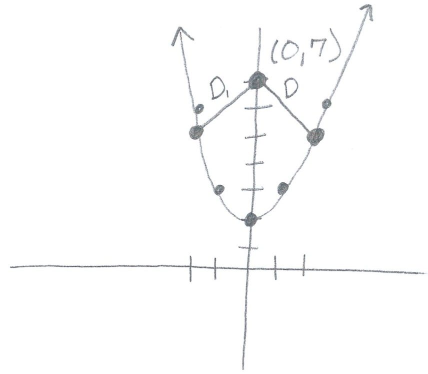
Check if max:
$P'' = -100 < 0$
max by 2nd DT

(5) Set $= 0$ and solve:

$$0 = 1950 - 100p$$
$$100p = 1950$$
$$p = 19.50$$

(6) Should charge $\boxed{\$19.50/\text{ream}}$

Ex. 4 ① Minimize distance:



The distance between any two points is

$$D^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2.$$

② Here, one of the points is on the graph:

$$(x_1, y_1) = (x, x^2+2).$$

The other point is $(x_2, y_2) = (0,7)$.

③ $D^2 = (x-0)^2 + ((x^2+2)-7)^2$

$D^2 = x^2 + (x^2-5)^2$

④ Take derivative:

$\frac{d}{dx}\left(D^2 = x^2 + (x^2-5)^2\right)$

$2D\frac{dD}{dx} = 2x + 2(x^2-5)(2x) = 4x^3 - 18x$

⑤ Set $\frac{dD}{dx} = 0$, so we are solving:

$2D(0) = 2x + 4x(x^2-5)$

$0 = 2x + 4x^3 - 20x$

$0 = 4x^3 - 18x$

$0 = \underbrace{2x}_{x=0}\underbrace{(2x^2-9)}_{x^2 = 9/2}$

$x = \pm\sqrt{\frac{9}{2}}$

* Here, it's important to use 1st DT or 2nd DT:

$D'' = 12x^2 - 18$

@ $x=0$: $D'' = -18 < 0$ ∩ max

@ $x = +\sqrt{\frac{9}{2}}$: $D'' = 12\left(\sqrt{\frac{9}{2}}\right)^2 - 18 = 12\left(\frac{9}{2}\right) - 18 > 0$ ∪ min

@ $x = -\sqrt{\frac{9}{2}}$: $D'' = 12\left(-\sqrt{\frac{9}{2}}\right)^2 - 18 = 12\left(\frac{9}{2}\right) - 18 > 0$ ∪ min

⑥ Find the y-coordinates using $y = x^2+2$:

$x = \sqrt{\frac{9}{2}}$

$y = \left(\sqrt{\frac{9}{2}}\right)^2 + 2$

$y = \frac{9}{2} + 2 = \frac{13}{2}$

$x = -\sqrt{\frac{9}{2}}$

$y = \left(-\sqrt{\frac{9}{2}}\right)^2 + 2$

$y = \frac{9}{2} + 2 = \frac{13}{2}$

$$\boxed{\left(\sqrt{\frac{9}{2}}, \frac{13}{2}\right) \quad \left(-\sqrt{\frac{9}{2}}, \frac{13}{2}\right)}$$

Ex.5 1) Minimize distance: $D^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$

2) The two points are $(0,2)$ and $(x,y) = (x, 7x+1)$.

3) $D^2 = (x-0)^2 + ((7x+1)-2)^2$

$D^2 = x^2 + (7x-1)^2$

4) Take derivative:

$2DD' = 2x + 2(7x-1)(7) = 2x + 14(7x-1) = 100x - 14$

5) Set $D' = 0$ and solve:

$2D(0) = 100x - 14$

$0 = 100x - 14$

$14 = 100x$

$\frac{14}{100} = x$

$\frac{7}{50} = x$

Check if this is where the min occurs.

2nd DT: $D'' = 100 > 0$ ∪ min

6) Find y-coordinate:

$y = 7x + 1$

$y = 7\left(\frac{7}{50}\right) + 1$

$y = \frac{49}{50} + \frac{50}{50} = \frac{99}{50}$

$\boxed{\left(\frac{7}{50}, \frac{99}{50}\right)}$