Lesson 27: Antiderivatives and Indefinite Integration

Def. If $F'(x) = f(x)$, then $F(x)$ is the **antiderivative** of $f(x)$.

We write $\int f(x)\, dx = F(x)$ which is read as "the integral of $f(x)$ with respect to x equals $F(x)$."

"Indefinite integration" refers to the final answer being a function. We will discuss "definite integrals" later in this course.

Ex.1 **(a)** Differentiate $f(x) = x^2 + 2$.

This type of problem is familiar to us. We know $f'(x) = 2x$

(b) Find $\int 2x\, dx$.

Integration is the opposite of differentiation. We look at $2x$ and have to think about what function has $2x$ as its derivative. As we can see from part (a), $2x$ is the derivative of x^2. However, since the derivative of a constant is 0, we don't know if $2x$ is the derivative of $x^2 + 2$, $x^2 - 1234$, $x^2 + \pi^e$, or x^2 plus any other constant. To take this into account, we use C to stand for any constant. This means $\int 2x\, dx = x^2 + C$.

Note: On quizzes, you will lose a point if you forget "+C " when answering a question on indefinite integration.

This is one example of integration. See the table of derivatives and integrals for the functions we know how to integrate right now. Similar to when we first learned derivatives, we do NOT know how to integrate products or quotients.

$$\int f'(x) \cdot g'(x)\, dx \neq f(x) \cdot g(x) + C$$

Those two are not equal because

$$\frac{d}{dx}\left[f(x) \cdot g(x)\right] \neq f'(x) \cdot g'(x)$$

Just as we need special rules to take the derivative of products, quotients, and compositions of functions (the product rule, quotient rule, and chain rule respectively), we will need special rules to integrate products, quotients, and compositions of functions. You will learn some of these in Calc 2.

Ex.2 Find $\int \left(\sqrt{x}+7\right)\ dx$.

$$\begin{aligned}\int \left(\sqrt{x}+7\right)\ dx &= \int \left(x^{1/2}+7\right)\ dx\\ &= \int x^{1/2}\ dx + \int 7\ dx\\ &= \frac{1}{\frac{1}{2}+1}x^{\frac{1}{2}+1} + 7x + C\\ &= \frac{1}{\frac{3}{2}}x^{3/2} + 7x + C\\ &= \frac{2}{3}x^{3/2} + 7x + C\\ &= \frac{2}{3}\sqrt{x^3} + 7x + C\end{aligned}$$

Ex.3 Find $\int \csc(x)\left(\cot(x)-\csc(x)\right)\ dx$.

This integral has a product that we don't know how to integrate. We need to rewrite it before we can do the integration.

$$\begin{aligned}\int \csc(x)\left(\cot(x)-\csc(x)\right)\ dx &= \int \left(\csc(x)\cot(x)-\csc^2(x)\right)\ dx\\ &= \int \csc(x)\cot(x)\ dx - \int \csc^2(x)\ dx\\ &= -\csc(x) - (-\cot(x)) + C\\ &= -\csc(x) + \cot(x) + C\end{aligned}$$

Ex.4 Find $\int \left(\frac{x^3+2}{4}\right)\ dx$.

This integral has a fraction, but since the denominator of the fraction is just 4 and there is no x to deal with, we can pull $\frac{1}{4}$ out front as a coefficient.

$$\begin{aligned}
\int \left(\frac{x^3+2}{4}\right)\ dx &= \int \frac{1}{4}\left(x^3+2\right)\ dx \\
&= \frac{1}{4}\int (x^3+2)\ dx \\
&= \frac{1}{4}\left(\int x^3\ dx + \int 2\ dx\right) \\
&= \frac{1}{4}\left(\frac{1}{3+1}x^{3+1} + 2x + C\right) \\
&= \frac{1}{4}\left(\frac{1}{4}x^4 + 2x + C\right) \\
&= \frac{1}{4}\cdot\frac{1}{4}x^4 + \frac{1}{4}\cdot 2x + \frac{1}{4}\cdot C \\
&= \frac{1}{16}x^4 + \frac{1}{2}x + C
\end{aligned}$$

Since C is a constant, we can write C instead of $\frac{C}{4}$. In general, we can just write the $+C$ after we've finished integrating.

Ex.5 Find $\int \left(\frac{x^4+\sqrt{x}}{3x}\right)\,dx$.

Since there's an x in the denominator this time, we can't bring the entire denominator out front. However, we *could* bring the $\frac{1}{3}$ out front as a coefficient. For this example, we can divide both terms in the numerator by the denominator.

$$\begin{aligned}
\int \left(\frac{x^4+\sqrt{x}}{3x}\right)\,dx &= \int \left(\frac{x^4}{3x}+\frac{\sqrt{x}}{3x}\right)\,dx \\
&= \int \left(\frac{x^3}{3}+\frac{x^{1/2}}{3x}\right)\,dx \\
&= \frac{1}{3}\int x^3\,dx + \frac{1}{3}\int x^{\frac{1}{2}-1}\,dx \\
&= \frac{1}{3}\int x^3\,dx + \frac{1}{3}\int x^{-\frac{1}{2}}\,dx \\
&= \frac{1}{3}\cdot\frac{1}{3+1}x^{3+1} + \frac{1}{3}\cdot\frac{1}{-\frac{1}{2}+1}x^{-\frac{1}{2}+1} + C \\
&= \frac{1}{3}\cdot\frac{1}{4}x^4 + \frac{1}{3}\cdot\frac{1}{\frac{1}{2}}x^{1/2} + C \\
&= \frac{1}{12}x^4 + \frac{1}{3}\cdot\frac{2}{1}x^{1/2} + C \\
&= \frac{1}{12}x^4 + \frac{2}{3}x^{1/2} + C
\end{aligned}$$

Ex.6 Find $\int \left(\cos(x)+\frac{1}{2x}+\sqrt[3]{x}+e^x+ex\right)\,dx$.

$$\begin{aligned}
&\int (\cos(x)+\frac{1}{2x}+\sqrt[3]{x}+e^x+ex)\,dx \\
&\quad= \int \cos(x)\,dx + \frac{1}{2}\int \frac{1}{x}\,dx + \int x^{1/3}\,dx + \int e^x\,dx + e\int x\,dx \\
&\quad= \sin(x) + \frac{1}{2}\ln\left(|x|\right) + \frac{1}{\frac{1}{3}+1}x^{\frac{1}{3}+1} + e^x + e\cdot\frac{1}{1+1}x^{1+1} + C \\
&\quad= \sin(x) + \frac{1}{2}\ln\left(|x|\right) + \frac{1}{\frac{4}{3}}x^{4/3} + e^x + e\cdot\frac{1}{2}x^2 + C \\
&\quad= \sin(x) + \frac{1}{2}\ln\left(|x|\right) + \frac{3}{4}x^{4/3} + e^x + \frac{e}{2}x^2 + C
\end{aligned}$$

We can see if we have done the integration correctly by checking that the derivative of $\left(\sin(x)+\frac{1}{2}\ln\left(|x|\right)+\frac{3}{4}x^{4/3}+e^x+\frac{e}{2}x^2+C\right)$ is $\left(\cos(x)+\frac{1}{2x}+\sqrt[3]{x}+e^x+ex\right)$.