Lesson 28: Indefinite Integration

When we integrate a function, we add "$+C$" for the general solution. If we're given a data point for the curve, we can solve for C and find the particular solution.

Ex.1 Given the initial value problem (IVP) $\begin{cases} y' = \frac{1}{x} \\ y(1) = 2 \end{cases}$, find $y(e^3)$.

Since $y' = \frac{1}{x}$, $y = \int y'\,dx = \int \frac{1}{x}\,dx = \ln|x| + C$.
Now we can use the data point $y(1) = 2$ to find C.

$$y(1) = \ln|1| + C = 2$$
$$0 + C = 2$$
$$C = 2$$

Then the particular solution is $y = \ln|x| + 2$.
Finally, we can find $y(e^3) = \ln|e^3| + 2$

$$= 3\ln|e| + 2$$
$$= 3 + 2 \quad = \boxed{5}$$

find the particular solution

Ex.2 Solve the IVP $\begin{cases} y'' = x - 2 \\ y'(0) = 3 \\ y(1) = 0 \end{cases}$.

First, we need to find y': $y' = \int y''\,dx = \int (x-2)\,dx$

$$y' = \frac{1}{2}x^2 - 2x + C$$

Use data point $y'(0) = 3$ to find C: $y'(0) = \frac{1}{2}(0)^2 - 2\cdot 0 + C = 3$

$$C = 3$$

Next, find y: $y = \int y'\,dx = \int (\frac{1}{2}x^2 - 2x + 3)\,dx$

$$= \frac{1}{2}\cdot\frac{1}{3}x^3 - 2\cdot\frac{1}{2}x^2 + 3x + C = \frac{1}{6}x^3 - x^2 + 3x + C$$

Use data point $y(1) = 0$ to find C: $y(1) = \frac{1}{6} - 1 + 3 + C = 0$

$$\frac{13}{6} + C = 0$$
$$C = -\frac{13}{6}$$

$$\boxed{y = \frac{1}{6}x^3 - x^2 + 3x - \frac{13}{6}} \Leftarrow$$

Ex. 3 Solve the IVP $\begin{cases} y' = \sin(x) - 2 \\ y(\frac{\pi}{3}) = 1 \end{cases}$.

$$y = \int y'\,dx = \int (\sin(x) - 2)\,dx = -\cos(x) - 2x + C$$

$$y(\tfrac{\pi}{3}) = -\cos(\tfrac{\pi}{3}) - 2\cdot\tfrac{\pi}{3} + C = 1$$

$$-\tfrac{1}{2} - \tfrac{2\pi}{3} + C = 1$$

$$C = 1 + \tfrac{1}{2} + \tfrac{2\pi}{3} = \tfrac{3}{2} + \tfrac{2\pi}{3}$$

$$\boxed{y = -\cos(x) - 2x + \tfrac{3}{2} + \tfrac{2\pi}{3}}$$

Ex. 4 Solve the IVP $\begin{cases} y'' = e^x + 2x \\ y'(1) = 1 \\ y(0) = -7 \end{cases}$.

$$y' = \int y''\,dx = \int (e^x + 2x)\,dx = e^x + x^2 + C$$

$$y'(1) = e^1 + 1^2 + C = 1$$

$$e + 1 + C = 1$$

$$C = -e \Rightarrow y' = e^x + x^2 - e$$

just a constant!

$$y = \int y'\,dx = \int (e^x + x^2 - e)\,dx = e^x + \tfrac{1}{3}x^3 - ex + C$$

$$y(0) = e^0 + \tfrac{1}{3}\cdot 0^3 - e\cdot 0 + C = -7$$

$$1 + 0 - 0 + C = -7$$

$$C = -8$$

$$\boxed{y = e^x + \tfrac{1}{3}x^3 - ex - 8}$$

Ex. 5 The rate of growth $\frac{dP}{dt}$ of the wraith population is proportional to the square of t with a constant coefficient of 3 where t is time in days $(0 \le t \le 20)$. The initial population is 1000. Find the population after 2 days.

- $\frac{dP}{dt}$ is proportional to $3t^2$, so $\frac{dP}{dt} = 3t^2$.
  $P = \int \frac{dP}{dt}\cdot dt = \int 3t^2\,dt = t^3 + C$.
- The initial population is 1000, so $P(0) = 1000$.
  $P(0) = 0^3 + C = 1000 \Rightarrow C = 1000$.

• $P(t) = t^3 + 1000$; $P(2) = ?$
$P(2) = 2^3 + 1000 = \boxed{1008}$

Ex. 6 Shawn and Jules are in a hot air balloon rising vertically with a velocity of 4.3 m/s. They drop grapes from the balloon when it is 341.4 m above the ground. Use $a(t) = -9.8 \text{ m/s}^2$ for acceleration due to gravity.

(a) After how many seconds will the grape hit the ground?

Recall: $a = v' = s''$, $v = s'$

The grape hits the ground when $s(t) = 0$, so we need to solve the IVP
$$\begin{cases} s''(t) = a(t) = -9.8 \\ s'(0) = v(0) = 4.3 \\ s(0) = 341.4 \end{cases}$$

(Use $t=0$ to represent the time the grape is dropped.)

$s'(t) = \int s''\,dt = \int -9.8\,dt = -9.8t + C$

$s'(0) = 0 + C = 4.3 \Rightarrow s'(t) = -9.8t + 4.3$

$s(t) = \int s'\,dt = \int(-9.8t + 4.3)\,dt = -4.9t^2 + 4.3t + C$

$s(0) = 0 + 0 + C = 341.4$

$s(t) = -4.9t^2 + 4.3t + 341.4 = 0$

$t = \dfrac{-4.3 \pm \sqrt{(4.3)^2 - 4(-4.9)(341.4)}}{2(-4.9)} \approx 8.80, \cancel{-7.92}$ ← negative time

$\boxed{t = 8.80 \text{ s}}$

(b) What is its velocity when it hits the ground? (Use t from part (a).)

$v(t) = s'(t) = -9.8t + 4.3$

$s'(8.80) = -9.8(8.80) + 4.3 = \boxed{-81.94 \text{ m/s}}$