Lesson 29: Area and Riemann Sums

* Summations

Ex. 1 Evaluate $\sum_{i=3}^{5} i(1-i^2)$.

Anatomy of a summation

$$\sum_{i=3}^{5} i(1-i^2)$$

5 ← The ending value, i.e. the last number we plug into the general term

capital sigma denotes we will add terms → $\sum$

$i(1-i^2)$ ← The general term. Plug the different numbers into all i's.

i is the index variable.

3 ← The starting value, i.e. the first number we plug into the general term.

To evaluate, we look at the starting and ending values for the index i. In the case, i goes from 3 to 5. This means we evaluate the general term at $i=3$, $i=4$, and $i=5$. Note that i is always an integer. Once we evaluate the general term for the different values of i, we add up the terms.

$$\begin{aligned}\sum_{i=3}^{5} i(1-i^2) &= \underbrace{3(1-3^2)}_{i=3} + \underbrace{4(1-4^2)}_{i=4} + \underbrace{5(1-5^2)}_{i=5} \\ &= 3(1-9) + 4(1-16) + 5(1-25) \\ &= 3(-8) + 4(-15) + 5(-24) \\ &= (-24) + (-60) + (-120) \\ &= \boxed{-204}\end{aligned}$$

Ex. 2 Use sigma notation to write the sum for $i = 1, 2, \ldots, n$:

$$(1-1)^2 + (4-2)^2 + (9-3)^2 + \ldots + (n^2-n)^2.$$

The "..." means for all integers between 2 and n.

Since $i = 1, 2, \ldots, n$ we know the starting and ending values for the index, so we'll have $\sum_{i=1}^{n} \underline{\text{general term}}$. Now, we just have to find the general term.

When the ending value is n, this is easy because the last term $(n^2-n)^2$ is when $i=n$. All we have to do is write i where we see n, so the sigma notation for the sum is $\boxed{\sum_{i=1}^{n} (i^2-i)^2}$.

We can check this with the given sum:

$$\underbrace{(1-1)^2}_{i=1} + \underbrace{(4-2)^2}_{i=2} + \underbrace{(9-3)^2}_{i=3} + \ldots + \underbrace{(n^2-n)^2}_{i=n}$$

$$(1^2-1)^2 + (2^2-2)^2 + (3^2-3)^2 + \ldots + (n^2-n)^2 \quad \checkmark$$

Ex.3 Use sigma notation to write the sum for $i = 1, 2, \ldots, 7$:

$$2^1 + 3^2 + \ldots + 8^7$$

Here, the summation doesn't stop at n, so finding the general term is a little trickier.
We can think of it this way:

$$\underbrace{2^1}_{i=1} + \underbrace{3^2}_{i=2} + \ldots + \underbrace{8^7}_{i=7}$$

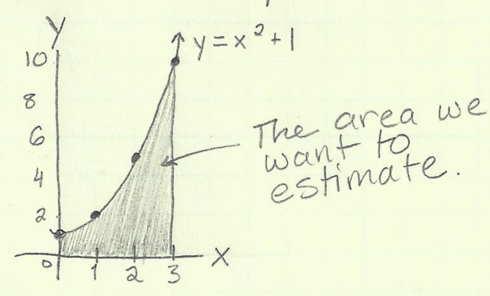
When $i = 1$, we have 2^1, and $2 = 1+1 = i+1$, so we could write $(i+1)^i$. Will this hold for the other terms?
When $i=2$: $(2+1)^2 = 3^2$ ✓
$i=7$: $(7+1)^7 = 8^7$ ✓
This means $(i+1)^i$ is the general term, so the summation is $\boxed{\sum_{i=1}^{7} (i+1)^i}$

* Riemann Sums

We use Riemann Sums to estimate the area under a curve with rectangles. These problems are usually phrased as "use the Left and Right Riemann sums with n rectangles to estimate the (signed) area under the curve $y = f(x)$ on the interval $[a, b]$."

To make sense of this, we'll work through an example.

Ex.4 Use the Left and Right Riemann Sums with 3 rectangles to estimate the area under the curve of $y = x^2+1$ on the interval $[0, 3]$.



Let's start by graphing $y = x^2+1$:

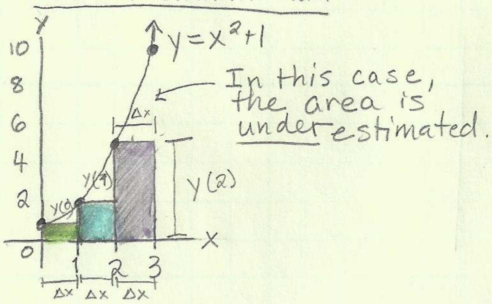
We want to draw 3 rectangles near this curve. The bottom of the rectangles will sit on the interval $[0, 3]$. For Riemann Sums, we want each rectangle to have the same width Δx. We can find this width by dividing the length of the interval into 3 pieces, so

$$\Delta x = \frac{3-0}{3} = \frac{3}{3} = 1.$$

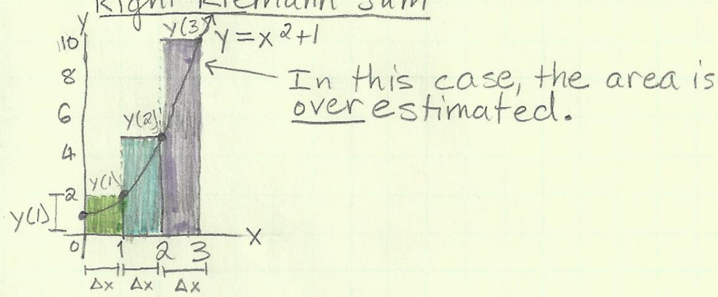
This means the base of the 1st rectangle is $[0, 1]$, the base of the 2nd is $[1, 2]$, and the 3rd is $[2, 3]$.
Now, we have to find the height of each rectangle. For Riemann Sums, we use the value of the curve at either the left or right endpoint of the base for each rectangle.

This means the height of the 1^{st} is $y(0)$ or $y(1)$, the height of the 2^{nd} is $y(1)$ or $y(2)$, and the height of the 3^{rd} is $y(2)$ or $y(3)$.
When we pick the left endpoints to find the height, we have the Left Riemann Sum. When we use the right end points, we have the Right Riemann Sum. The two pictures are below

Left Riemann Sum



Right Riemann Sum



Now we need to add up the area of the different rectangles to estimate the area under $y=x^2+1$ on $[0,3]$

$$\text{Left Riemann Sum (LRS)} = \overset{\text{width}\cdot\text{height}}{\Delta x\cdot y(0)} + \overset{\text{width}\cdot\text{height}}{\Delta x\cdot y(1)} + \overset{\text{width}\cdot\text{height}}{\Delta x\cdot y(2)}$$

(Note: $= \Delta x\,[y(0)+y(1)+y(2)]$)

$$\begin{aligned}
&= 1\cdot(0^2+1) + 1\cdot(1^2+1) + 1\cdot(2^2+1)\\
&= 1\cdot(1) + 1\cdot(2) + 1\cdot(5)\\
&= 1+2+5\\
&= \boxed{8}
\end{aligned}$$

$$\begin{aligned}
\text{Right Riemann Sum (RRS)} &= \Delta x\cdot y(1) + \Delta x\cdot y(2) + \Delta x\cdot y(3)\\
&= 1\cdot(1^2+1) + 1\cdot(2^2+1) + 1\cdot(3^2+1)\\
&= 1\cdot(2) + 1\cdot(5) + 1\cdot(10)\\
&= 2+5+10\\
&= \boxed{17}
\end{aligned}$$

Now that we've seen an example, let's look at the formulas to calculate Riemann Sums.

* In general, to use Left and Right Riemann Sums with n rectangles to estimate the area under the curve of $y=f(x)$ on the interval of $[a,b]$, we have

$$\Delta x = \frac{b-a}{n}$$

$x_i = a + i\cdot\Delta x$, $\quad i = 0, 1, 2, \ldots, n-1, n$ is the index for the list of endpoints

$$LRS = \sum_{i=0}^{n-1} \Delta x\cdot f(x_i) \quad \left(= \Delta x\cdot\sum_{i=0}^{n-1} f(x_i)\right)$$

$$RRS = \sum_{i=1}^{n} \Delta x\cdot f(x_i) \quad \left(= \Delta x\cdot\sum_{i=1}^{n} f(x_i)\right)$$

Calculate the summation, then multiply the result by Δx.

Ex.4 (Again)
Use the LRS and RRS with 3 rectangles to estimate the area under the curve $y = x^2+1$ on the interval $[0, 3]$. Write the answer in sigma notation.

We have $n=3$, $f(x) = x^2+1$, and $[a,b] = [0,3]$.
Using the formulas, $\Delta x = \frac{b-a}{n} = \frac{3-0}{3} = 1$

$$x_i = a + i \cdot \Delta x = 0 + i \cdot 1 = i$$

$$f(x_i) = (x_i)^2 + 1 = i^2 + 1$$

OR

$$f(i) = i^2 + 1$$

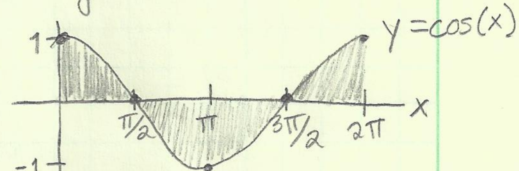
same answer.

Then $LRS = \sum_{i=0}^{n-1} \Delta x \cdot f(x_i) = \sum_{i=0}^{3-1} 1 \cdot (i^2+1) = \boxed{\sum_{i=0}^{2} (i^2+1)}$

$RRS = \sum_{i=1}^{n} \Delta x \cdot f(x_i) = \sum_{i=1}^{3} 1 \cdot (i^2+1) = \boxed{\sum_{i=1}^{3} (i^2+1)}$

*Note: The general term for the Left and Right Riemann Sums is the same. Only the starting and ending values for the index change.

Also, $y = f(x)$ can be negative, as with $y = \cos(x)$. That's okay because we look at the signed area. Basically, we're look at the net are above the x-axis, so when the curve is below the x-axis, we'll treat it like negative area.



Ex.5 Use the LRS and RRS with 200 rectangles to estimate the area under $y = \sqrt{x} + 2$ on the interval $[0, 100]$ (in sigma notation).

$n = 200$
$[a,b] = [0, 100]$
$f(x) = \sqrt{x} + 2$

$$\Delta x = \frac{100-0}{200} = \frac{100}{200} = \frac{1}{2}$$

$$x_i = 0 + i \cdot \frac{1}{2} = \frac{i}{2}$$

$$f(x_i) = f\left(\frac{i}{2}\right) = \sqrt{\frac{i}{2}} + 2$$

$$LRS = \sum_{i=0}^{200-1} \left(\tfrac{1}{2}\right)\left(\sqrt{\tfrac{i}{2}} + 2\right) = \boxed{\sum_{i=0}^{199} \left(\tfrac{1}{2}\right)\left(\sqrt{\tfrac{i}{2}} + 2\right)}$$

$$\left(= \frac{1}{2} \sum_{i=0}^{199} \left(\sqrt{\tfrac{i}{2}} + 2\right)\right)$$

$$RRS = \boxed{\sum_{i=1}^{200} \left(\tfrac{1}{2}\right)\left(\sqrt{\tfrac{i}{2}} + 2\right)}$$

$$\left(= \frac{1}{2} \sum_{i=1}^{200} \left(\sqrt{\tfrac{i}{2}} + 2\right)\right)$$

Ex.6 Use the LRS and RRS with 20 rectangles to estimate the area under $y = 3\cos(2x)$ on the interval $[\pi, 2\pi]$ (in sigma notation).

$$\left.\begin{array}{l} n = 20 \\ [a,b] = [\pi, 2\pi] \\ f(x) = 3\cos(2x) \end{array}\right\} \quad \begin{array}{l} \Delta x = \frac{2\pi - \pi}{20} = \frac{\pi}{20} \\ x_i = \pi + i \cdot \frac{\pi}{20} \end{array}$$

$$f(x_i) = f\left(\pi + i \cdot \tfrac{\pi}{20}\right) = 3\cos\left(2\left(\pi + \tfrac{i\pi}{20}\right)\right)$$

$$LRS = \sum_{i=0}^{20-1} \left(\frac{\pi}{20}\right)\left(3\cos\left(2\left(\pi + \frac{i\pi}{20}\right)\right)\right)$$
$$= \boxed{\frac{\pi}{20} \sum_{i=0}^{19} 3\cos\left(2\left(\pi + \frac{i\pi}{20}\right)\right)}$$
$$\left(= \frac{3\pi}{20} \sum_{i=0}^{19} \cos\left(2\left(\pi + \frac{i\pi}{20}\right)\right)\right)$$

$$RRS = \sum_{i=1}^{20} \left(\frac{\pi}{20}\right)\left(3\cos\left(2\left(\pi + \frac{i\pi}{20}\right)\right)\right)$$
$$= \boxed{\frac{\pi}{20} \sum_{i=1}^{20} 3\cos\left(2\left(\pi + \frac{i\pi}{20}\right)\right)}$$
$$\left(= \frac{3\pi}{20} \sum_{i=1}^{20} \cos\left(2\left(\pi + \frac{i\pi}{20}\right)\right)\right)$$

Ex.7 Use the LRS and RRS with 3 rectangles to estimate the area under the curve $y = e^x$ on the interval $[2, 7]$.

$$\left.\begin{array}{l} n = 3 \\ [a,b] = [2, 7] \end{array}\right\} \quad \Delta x = \frac{7-2}{3} = \frac{5}{3}$$

$$x_i = 2 + \tfrac{5}{3} \cdot i \quad , i = 0, 1, 2, 3$$

Method 1: Make a chart with i, x_i, and $f(x_i)$.

i	$x_i = 2 + \frac{5}{3} \cdot i$	$f(x_i)$	
0	$2 + \frac{5}{3} \cdot 0 = 2$	$f(2) = e^2$	$= f(x_0)$
1	$2 + \frac{5}{3} \cdot 1 = \frac{6}{3} + \frac{5}{3} = \frac{11}{3}$	$f(\frac{11}{3}) = e^{11/3}$	$= f(x_1)$
2	$2 + \frac{5}{3} \cdot 2 = \frac{6}{3} + \frac{10}{3} = \frac{16}{3}$	$f(\frac{16}{3}) = e^{16/3}$	$= f(x_2)$
3	$2 + \frac{5}{3} \cdot 3 = 2 + 5 = 7$	$f(7) = e^7$	$= f(x_3)$

$$LRS = \sum_{i=0}^{2} \Delta x \cdot f(x_i)$$
$$= \frac{5}{3} \cdot \sum_{i=0}^{2} f(x_i)$$
$$= \frac{5}{3}\left[f(x_0) + f(x_1) + f(x_2)\right]$$
$$= \frac{5}{3}\left[e^2 + e^{11/3} + e^{16/3}\right]$$
$$\approx \frac{5}{3}\left[253.64\right]$$
$$\approx \boxed{422.73}$$

$$RRS = \sum_{i=1}^{3} \frac{5}{3} \cdot f(x_i)$$
$$= \frac{5}{3} \cdot \sum_{i=1}^{3} f(x_i)$$
$$= \frac{5}{3}\left[f(x_1) + f(x_2) + f(x_3)\right]$$
$$= \frac{5}{3}\left[e^{11/3} + e^{16/3} + e^7\right]$$
$$\approx \frac{5}{3}\left[1342.88\right]$$
$$\approx \boxed{2238.14}$$

Method 2: Find $f(x_i)$ in terms of i.

$$f(x_i) = f\left(2 + \tfrac{5}{3} \cdot i\right) = e^{2 + \frac{5}{3} \cdot i}$$

$$LRS = \frac{5}{3} \sum_{i=0}^{2} f(x_i)$$
$$= \frac{5}{3} \sum_{i=0}^{2} \left(e^{2 + 5/3 \, i}\right)$$
$$= \frac{5}{3}\left[e^{2 + \frac{5}{3} \cdot 0} + e^{2 + \frac{5}{3} \cdot 1} + e^{2 + \frac{5}{3} \cdot 2}\right]$$
$$= \frac{5}{3}\left[e^2 + e^{11/3} + e^{16/3}\right]$$
$$\approx \boxed{422.73}$$

$$RRS = \frac{5}{3} \sum_{i=1}^{3} f(x_i)$$
$$= \frac{5}{3} \sum_{i=1}^{3} e^{2 + \frac{5}{3} \cdot i}$$
$$= \frac{5}{3}\left[e^{2 + \frac{5}{3} \cdot 1} + e^{2 + \frac{5}{3} \cdot 2} + e^{2 + \frac{5}{3} \cdot 3}\right]$$
$$= \frac{5}{3}\left[e^{11/3} + e^{16/3} + e^7\right]$$
$$\approx \boxed{2238.14}$$