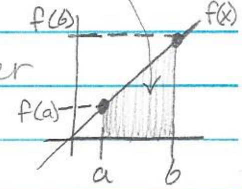
Lesson 30: Definite Integrals

$\int_a^b f(x)dx$



Def. definite integral $= \int_a^b f(x)dx$ is the area under the curve $f(x)$ on the interval $[a, b]$.

We approximated this area using Riemann Sums with n rectangles. As we increase the number of rectangles for the sums, we get a better approximation.

$n=3$ $\quad$ $n=6$ $\quad$ $n=18$

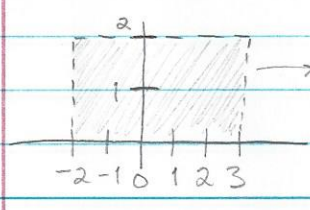
Shaded area is overestimation.

since $x_0 = a$ and $x_n = b$

(We write $\lim_{n\to\infty} \sum_{i=0}^{n} f(x_i)\Delta x = \int_a^b f(x)\,dx$.)

For the following definite integrals, evaluate by using geometric formulas.

Ex.1 $\int_{-2}^{3} 2\,dx = ?$

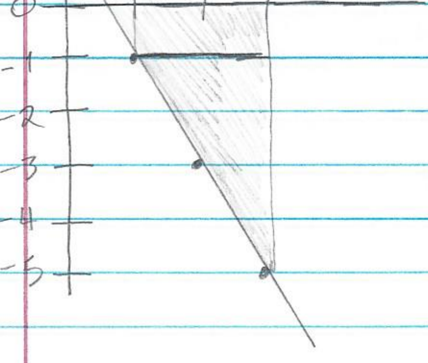


$\rightarrow$ Area of rectangle = length × width

$= (5) \times (2)$

$= \boxed{10}$

Ex.2 $\int_1^3 (-2x+1)\,dx = ?$



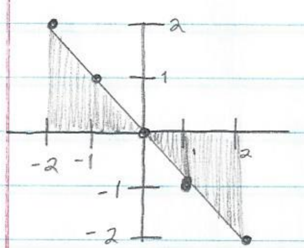
Treat as 2 regions: Rectangle and Triangle. Since it's below the x-axis, we use negative sign.

Area of rectangle: $-(1\cdot 2) = -2$

Area of triangle: $-\frac{1}{2}(2)(4) = -4$

Total area $= -2 + (-4) = \boxed{-6}$

Ex.3 $\int_{-2}^{2}(-x)\,dx = ?$



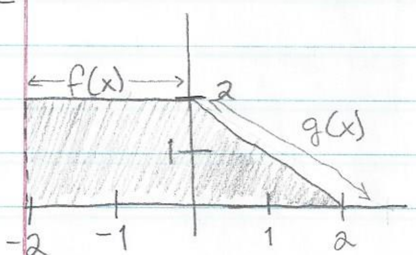
Break into positive and negative parts:

Positive Area (on $[-2,0]$): $\frac{1}{2}(2)(2) = 2$

Negative Area (on $[0,2]$): $-\frac{1}{2}(2)(2) = -2$

Total Area: $2 + (-2) = \boxed{0}$

Ex.4 Find the area of the shaded region.



Area of rectangle: $2 \times 2 = 4$

Area of triangle: $\frac{1}{2}(2)(2) = 2$

Total Area $= 4 + 2 = \boxed{6}$

Note: We can write this as a definite integral, but we'd have to use two different functions $f(x) = 2$ and $g(x) = -x + 2$.

Area $= \int_{-2}^{0} 2\,dx + \int_{0}^{2}(-x+2)\,dx$.

* **Properties of Definite Integrals**

$\int_a^a f(x)\,dx = 0$

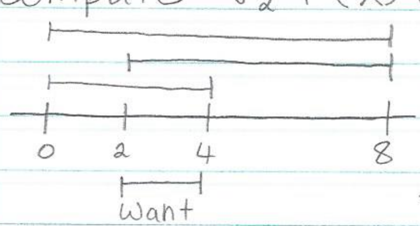
$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$

$\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$

Ex.5 Given $\int_0^4 f(x)\,dx = 7$, $\int_0^8 f(x)\,dx = -1$, and $\int_2^8 f(x)\,dx = -5$, compute $\int_2^4 f(x)\,dx$.



$\int_0^8 - \int_2^8 = \int_0^2$

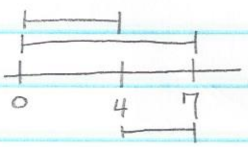
$\int_0^4 - \int_0^2 = \int_2^4$

$\Rightarrow \int_2^4 = \int_0^4 - \left(\int_0^8 - \int_2^8\right) = \int_0^4 - \int_0^8 + \int_2^8$

$$\int_2^4 f(x)dx = \int_0^4 f(x)\,dx - \int_0^8 f(x)dx + \int_2^8 f(x)dx$$
$$= 7-(-1)+(-5)$$
$$= 7+1-5 = \boxed{3}$$

Ex.6 Given $\int_0^7 f(x)dx = 1$, $\int_0^4 f(x) = -3$, and $\int_7^4 g(x)dx = 3$, compute $\int_4^7 [2f(x) - 3g(x)]\,dx$.

$$\int_4^7 [2f(x)-3g(x)]\,dx = \int_4^7 2f(x)\,dx - \int_4^7 3g(x)dx$$
$$= 2\int_4^7 f(x)dx - 3\int_4^7 g(x)dx$$



$$\int_4^7 = \int_0^7 - \int_0^4$$
$$\int_4^7 f(x)dxdx = \int_0^7 f(x)\,dx - \int_0^4 f(x)dx$$
$$= 1-(-3)$$
$$= 1+3 = 4$$
$$\int_4^7 g(x)dx = -\int_7^4 g(x)dx = -3$$

$$\int_4^7 [2f(x) - 3g(x)]dx = 2(4) - 3(-3) = 8+9 = \boxed{17}$$

Ex.7 Given $\int_{-3}^1 x^2 dx = \frac{28}{3}$, $\int_{-3}^1 3x\,dx = -12$, and $\int_{-3}^1 2\,dx = 8$, compute $\int_{-3}^1 (\frac{3}{2}x^2 + 2x - 4)\,dx$.

$$\int_{-3}^1 (\tfrac{3}{2}x^2 + 2x - 4)dx = \int_{-3}^1 \tfrac{3}{2}x^2\,dx + \int_{-3}^1 2x dx + \int_{-3}^1 (-4)dx$$
$$= \tfrac{3}{2}\int_{-3}^1 x^2 dx + 2\int_{-3}^1 x\,dx - 4\int_{-3}^1 1dx$$

$$\int_{-3}^1 3x dx = 3\int_{-3}^1 x dx \Rightarrow \int_{-3}^1 x dx = \tfrac{1}{3}\int_{-3}^1 3x\,dx = \tfrac{1}{3}(-12) = -4$$

$$\int_{-3}^1 2\,dx = 2\int_{-3}^1 1dx \Rightarrow \int_{-3}^1 1dx = \tfrac{1}{2}\int_{-3}^1 2dx = \tfrac{1}{2}(8) = 4$$

$$\int_{-3}^1 (\tfrac{3}{2}x^2 + 2x - 4)dx = \tfrac{\cancel{3}}{2}(\tfrac{28}{\cancel{3}}) + 2(-4) - 4(4)$$
$$= 14 - 8 - 16$$
$$= \boxed{-10}$$