Lesson 31: Fundamental Theorem of Calculus

* If $\int f(x)\,dx = F(x)$, $\int_a^b f(x)\,dx = F(b) - F(a)$.

Ex.1 $\int_2^7 x\,dx = ?$

$$\int x\,dx = \frac{1}{2}x^2 + C$$
$$\int_2^7 x\,dx = \left(\frac{1}{2}(7)^2 + C\right) - \left(\frac{1}{2}(2)^2 + C\right)$$
$$= \frac{49}{2} + C - \frac{4}{2} - C$$
$$= \boxed{\frac{45}{2}}$$

← Since the "+C" will always cancel, we don't need to include it.

Ex.2 $\int_1^4 \frac{\sqrt{x} - 1}{x}\,dx = ?$

$$\int_1^4 \left(\frac{\sqrt{x}}{x} - \frac{1}{x}\right)dx = \int_1^4 \left(x^{-1/2} - \frac{1}{x}\right)dx$$
$$= 2x^{1/2} - \ln|x| \Big|_{x=1}^{x=4}$$
$$= 2(4)^{1/2} - \ln|4| - \left(2(1)^{1/2} - \ln|1|\right)$$
$$= 4 - \ln(4) - 2$$
$$= \boxed{2 - \ln(4)}$$

Ex.3 $\int_{-1}^0 (2e^x + x)\,dx = ?$

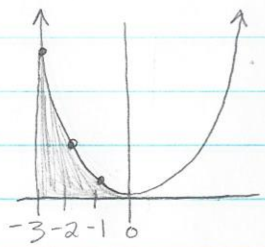
$$\int_{-1}^0 (2e^x + x)\,dx = 2e^x + \frac{1}{2}x^2 \Big|_{-1}^{0}$$
$$= 2e^0 + \frac{1}{2}(0)^2 - \left(2e^{-1} + \frac{1}{2}(-1)^2\right)$$
$$= 2 - \frac{2}{e} - \frac{1}{2}$$
$$= \boxed{\frac{3}{2} - \frac{2}{e}}$$

Ex.4 $\int_0^{\pi/3} \frac{1}{2}(\cos(x) - \sin(x))\,dx = ?$

$$\frac{1}{2}\int_0^{\pi/3} (\cos(x) - \sin(x))\,dx = \frac{1}{2}(\sin(x) + \cos(x))\Big|_0^{\pi/3}$$
$$= \frac{1}{2}\left[\sin\left(\frac{\pi}{3}\right) + \cos\left(\frac{\pi}{3}\right) - (\sin(0) + \cos(0))\right]$$
$$= \frac{1}{2}\left[\frac{\sqrt{3}}{2} + \frac{1}{2} - 1\right] = \frac{1}{2}\left(\frac{\sqrt{3} - 1}{2}\right)$$
$$= \boxed{\frac{\sqrt{3} - 1}{4}}$$

Ex.5 Find the area of the region bounded by

$y = x^2$, $y = 0$, $x = -3$ and $x = 0$.



$$\text{Area} = \int_{-3}^{0} x^2\,dx \quad \text{(the area under } x^2 \text{ on } [-3,0])$$
$$= \frac{1}{3}x^3 \Big|_{-3}^{0}$$
$$= \frac{1}{3}(0)^3 - \frac{1}{3}(-3)^3$$
$$= -\frac{1}{3}(-27) = \boxed{9}$$

Ex.6 $\int_{\pi/4}^{\pi/2} \csc(x)(\cot(x) - \csc(x))\,dx = ?$

$$\int_{\pi/4}^{\pi/2} (\csc(x)\cot(x) - \csc^2(x))\,dx = -\csc(x) + \cot(x) \Big|_{\pi/4}^{\pi/2}$$
$$= -\frac{1}{\sin(x)} + \frac{\cos(x)}{\sin(x)} \Big|_{\pi/4}^{\pi/2}$$
$$= -\frac{1}{\sin(\frac{\pi}{2})} + \frac{\cos(\pi/2)}{\sin(\pi/2)} - \left(-\frac{1}{\sin(\pi/4)} + \frac{\cos(\pi/4)}{\sin(\pi/4)}\right)$$
$$= -1 + 0 + \frac{1}{1/\sqrt{2}} - 1 = \boxed{-2 + \sqrt{2}}$$

Ex.7 $\int_{\pi/2}^{\pi} \frac{\cot(x)+1}{\csc(x)}\,dx = ?$

$$\int_{\pi/2}^{\pi} \frac{1}{\csc(x)}(\cot(x)+1)\,dx = \int_{\pi/2}^{\pi} \sin(x)\left(\frac{\cos(x)}{\sin(x)} + 1\right)dx$$
$$= \int_{\pi/2}^{\pi} (\cos(x) + \sin(x))\,dx$$
$$= \sin(x) - \cos(x) \Big|_{\pi/2}^{\pi}$$
$$= \cancel{\sin(\pi)} - \cos(\pi) - (\sin(\tfrac{\pi}{2}) - \cancel{\cos(\tfrac{\pi}{2})})$$
$$= -(-1) - 1 = 1 - 1 = \boxed{0}$$

Ex.8 $\int_{-2}^{0} (2x-1)\,dx = ?$

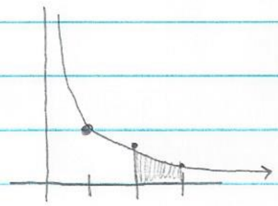
$$\int_{-2}^{0} (2x-1)\,dx = x^2 - x \Big|_{-2}^{0} = 0^2 - 0 - ((-2)^2 - (-2)) = -(4+2) = \boxed{-6}$$

Ex.9 $\int_{1}^{9} \frac{1}{2x}\,dx = ?$

$$\frac{1}{2}\int_{1}^{9} \frac{1}{x}\,dx = \frac{1}{2}\ln|x| \Big|_{1}^{9} = \frac{1}{2}(\ln(9) - \ln(1)) = \frac{1}{2}\ln(9) = \boxed{\ln(3)}$$

Ex.10 Find the area of the region bounded by

$y = \frac{1}{x} + 1$, $y = 0$, $x = 2$ and $x = 3$



$$\text{Area} = \int_2^3 \left(\frac{1}{x} + 1\right) dx$$
$$= \ln|x| + x \Big|_2^3$$
$$= \ln(3) + 3 - (\ln(2) + 2)$$
$$= \ln(3) + 3 - \ln(2) - 2$$
$$= \ln(3) - \ln(2) + 1 = \boxed{\ln\left(\frac{3}{2}\right) + 1}$$

Ex.11 $\int_{2\pi/3}^{3\pi/2} (2\sin(x) - 1)\, dx = ?$

$$\int_{2\pi/3}^{3\pi/2} (2\sin(x) - 1)\, dx = -2\cos(x) - x \Big|_{2\pi/3}^{3\pi/2}$$
$$= \cancel{-2\cos\left(\frac{3\pi}{2}\right)} - \frac{3\pi}{2} - \left(-2\cos\left(\frac{2\pi}{3}\right) - \frac{2\pi}{3}\right)$$
$$= -\frac{3\pi}{2} + 2\left(-\frac{1}{2}\right) + \frac{2\pi}{3}$$
$$= -\frac{9\pi}{6} + \frac{4\pi}{6} - 1 = \boxed{-\frac{5\pi}{6} - 1}$$

Ex.12 $\int_0^4 \frac{x + \sqrt{x}}{2}\, dx = ?$

$$\frac{1}{2}\int_0^4 (x + \sqrt{x})\, dx = \frac{1}{2}\left(\frac{1}{2}x^2 + \frac{2}{3}x^{3/2}\right) \Big|_{x=0}^{x=4}$$
$$= \frac{1}{2}\left[\frac{1}{2}(4^2) + \frac{2}{3}(4)^{3/2} - \left(\frac{1}{2}(0)^2 + \frac{2}{3}(0)^{3/2}\right)\right]$$
$$= \frac{1}{2}\left[8 + \frac{2}{3} \cdot 8\right]$$
$$= 4 + \frac{8}{3} = \boxed{\frac{20}{3}}$$

Ex.13 $\int_{-1}^{2} \left(\frac{x^3 + 1}{x^2}\right) dx$

$$\int_{-1}^{2} \left(\frac{x^3}{x^2} + \frac{1}{x^2}\right) dx = \int_{-1}^{2} \left(x + \frac{1}{x^2}\right) dx = \frac{1}{2}x^2 - \frac{1}{x} \Big|_{-1}^{2}$$
$$= \frac{1}{2}(2)^2 - \frac{1}{2} - \left(\frac{1}{2}(-1)^2 - \frac{1}{(-1)}\right)$$
$$= 2 - \frac{1}{2} - \left(\frac{1}{2} + 1\right)$$
$$= 2 - \frac{1}{2} - \frac{1}{2} - 1 = 1 - 1 = \boxed{0}$$