Lesson 32: Fundamental Theorem of Calculus (Applications)

When given the rate of change of a function, we can integrate to find the general solution for the function. For indefinite integrals, if we are given an initial value, we can find the particular solution, i.e. the exact function. Then we can evaluate the function for any input.

If we are not given an initial condition, we can't find the exact function, but we can still find the **change in the function** between any two inputs by using the fundamental theorem of calculus.

In general, given the rate of change $f'(x)$, we can find the change in $f(x)$ between $x = a$ and $x = b$, by finding $f(b) - f(a) = \int_a^b f'(x)\,dx$.

Ex. 1 The growth rate of the Milky Way Replicator population is given by $P'(t) = 4000e^t + 300t^2$ where t is time in days. By how much does the population increase from $t = 0$ days to $t = 7$ days?

The population when $t=7$ is $P(7)$.
The population when $t=0$ is $P(0)$.
Then the increase in the population from $t=0$ to $t=7$ is $P(7) - P(0) = \int_0^7 P'(t)\,dt$

$$= \int_0^7 (4000e^t + 300t^2)\,dt$$

$$= 4000e^t + 300 \cdot \frac{1}{2+1} t^{2+1} \Big|_0^7$$

$$= 4000e^t + 100t^3 \Big|_0^7$$

$$= 4000e^7 + 100(7)^3 - [4000e^0 + 100(0)^3]$$

$$= 4000e^7 + 100(343) - [4000 + 0]$$

$$= 4000e^7 + 34300 - 4000$$

$$= \boxed{4000e^7 + 30300}$$

Ex. 2 An X-302 flies back and forth along a straight line with velocity (in m/s) given by

$$v(t) = 3\sqrt{t} - 7.$$

(a) Find the displacement between $t = 4$ seconds and $t = 9$ seconds.
(b) Find the time when the displacement is 0 after the X-302 takes off.
(c) Find the time when the acceleration is $0.75\ \mathrm{m/s^2}$ after the X-302 takes off.

(a) Displacement is the change in position, so we want to find

$$s(9) - s(4) = \int_4^9 s'(t)\,dt = \int_4^9 v(t)\,dt$$
$$= \int_4^9 (3\sqrt{t} - 7)\,dt$$
$$= \int_4^9 (3t^{1/2} - 7)\,dt$$
$$= 3 \cdot \frac{1}{\frac{1}{2}+1} t^{\frac{1}{2}+1} - 7t \Big|_4^9$$
$$= 3 \cdot \frac{1}{3/2} t^{3/2} - 7t \Big|_4^9$$
$$= 3 \cdot \frac{2}{3} t^{3/2} - 7t \Big|_4^9$$
$$= 2t^{3/2} - 7t \Big|_4^9$$
$$= 2(9)^{3/2} - 7(9) - [2(4)^{3/2} - 7(4)]$$
$$= 2(3)^3 - 63 - [2(2)^3 - 28]$$
$$= 2(27) - 63 - [2(8) - 28]$$
$$= 54 - 63 - [16 - 28]$$
$$= -9 - [-12]$$
$$= -9 + 12 = \boxed{3} \text{ meters}$$

Note: $9^{3/2} = (9^{1/2})^3 = 3^3$ and $4^{3/2} = (4^{1/2})^3 = 2^3$

(b) Displacement is 0 when it returns to the initial position. The initial position is the position at $t = 0$, so $s(0)$.
→ We want to find $t = T$ so that $0 = s(T) - s(0)$.

The time T when the displacement is 0.

$$0 = s(T) - s(0) = \int_0^T s'(t)\,dt = \int_0^T v(t)\,dt = \int_0^T (3\sqrt{t} - 7)\,dt$$
$$= 2t^{3/2} - 7t \Big|_0^T \text{ (from part (a))}$$
$$= 2T^{3/2} - 7T - [\cancel{2(0)^{3/2}}^{\,0} - \cancel{7(0)}^{\,0}]$$
$$= 2T^{3/2} - 7T$$

Now, we need to solve $0 = 2T^{3/2} - 7T$ for T.

$$2T^{3/2} - 7T = 0$$
$$2T^1 \cdot T^{1/2} - 7T = 0$$
$$T(2T^{1/2} - 7) = 0$$

$\Rightarrow$ ~~$T = 0$~~ and $2T^{1/2} - 7 = 0$

want time after it takes off, so $t > 0$.

$$2T^{1/2} = 7$$
$$T^{1/2} = \frac{7}{2}$$
$$T = \left(\frac{7}{2}\right)^2$$
$$\boxed{T = \frac{49}{4}} \text{ seconds}$$

(c) We want the time when the acceleration is $0.75 = \frac{3}{4}\ m/s^2$.

$a(t) = v'(t)$, so $a(t) = \frac{d}{dt}(3t^{1/2} - 7)$

$$= 3 \cdot \frac{1}{2} t^{1/2 - 1}$$
$$= \frac{3}{2} t^{-1/2}$$
$$= \frac{3}{2\sqrt{t}}.$$

Now, we have to solve $\frac{3}{4} = \frac{3}{2\sqrt{t}}$ for t.

$$\frac{3}{4} = \frac{3}{2\sqrt{t}} \quad \text{(cross multiply)}$$
$$6\sqrt{t} = 12$$
$$\sqrt{t} = 2$$
$$\boxed{t = 4} \text{ seconds.}$$

Ex. 3

~~Ex. 4~~ A puddle jumper slows to a stop with acceleration $a(t) = -(2t+1)^2$ mph per second, where t is the number of seconds since the pilot began decelerating. What is the decrease in velocity 5 s after deceleration begins?

Deceleration begins at $t=0$, so the change in velocity 5s after acceleration begins is

$$v(5)-v(0) = \int_0^5 v'(t)\,dt = \int_0^5 a(t)\,dt$$

$$= \int_0^5 -(2t+1)^2\,dt$$

can't undo chain rule yet, so we have to multiply it out

$$= -\int_0^5 (2t+1)(2t+1)\,dt$$

$$= -\int_0^5 (4t^2+2t+2t+1)\,dt$$

$$= -\int_0^5 (4t^2+4t+1)\,dt$$

$$= -\left(4\cdot\frac{1}{2+1}t^{2+1} + 4\cdot\frac{1}{1+1}t^{1+1} + t\right)\Big|_0^5$$

$$= -\left(\frac{4}{3}t^3 + 2t^2 + t\right)\Big|_0^5$$

$$= -\left(\frac{4}{3}(5)^3 + 2(5)^2 + 5\right) - \left[-\left(\frac{4}{3}(0)^3 + 2(0)^2 + 0\right)\right]$$

$$= -\left(\frac{4}{3}(125) + 2(25) + 5\right) - [-0]$$

$$= -\left(\frac{500}{3} + 50 + 5\right)$$

$$= -\left(\frac{500}{3} + 55\right)$$

$$= -\left(\frac{500}{3} + \frac{165}{3}\right)$$

$$= -\left(\frac{665}{3}\right)$$

$$= -\frac{665}{3}$$

The negative sign tells us the velocity decreased. Then the decrease in velocity is $\boxed{\frac{665}{3}}$ m/s.

Ex. 4

initially empty

~~Ex. 3~~ The Wet Bandits turn on a kitchen faucet at 9:00 pm and water flows into the sink at a rate of $r(t) = 8\sqrt[3]{t}$ where t is time in hours after 9:00 pm and $r(t)$ is in cubic feet per hour.

(b) If the sink holds 2 cubic feet of water, at what time will the sink be completely full?

(a) How much water flows from the faucet from midnight to 9:00 am?

(a) At midnight, $t = 3$ and at 9:00 am, $t = 12$.
We can say $R(t)$ is the amount of water (in cubic feet) that has flown out of the faucet t hours after 9:00 pm, i.e. $R(t) = \int r(t)\,dt$.
Then we want to find

$$R(12) - R(3) = \int_3^{12} r(t)\,dt = \int_3^{12} 8\sqrt[3]{t}\,dt$$

$$= 8\int_3^{12} t^{1/3}\,dt$$

$$= 8\,\frac{1}{\frac{1}{3}+1}\,t^{\frac{1}{3}+1}\Big|_3^{12}$$

$$= 8\,\frac{1}{\frac{4}{3}}\,t^{4/3}\Big|_3^{12}$$

$$= \overset{2}{\cancel{8}}\cdot\frac{3}{\cancel{4}}\,t^{4/3}\Big|_3^{12}$$

$$= 6t^{4/3}\Big|_3^{12}$$

$$= 6(12)^{4/3} - 6(3)^{4/3}$$

$$\approx \boxed{138.88}\ \text{ft}^3$$

(b) The sink is initially empty, so we know $R(0) = 0$.
This means we can find our particular $R(t)$ by solving the IVP $\begin{cases} R'(t) = r(t) = 8\sqrt[3]{t} \\ R(0) = 0 \end{cases}$.

Then we solve $2 = R(t)$ for t.

① $R(t) = \int 8\sqrt[3]{t}\,dt$

$= 8\int t^{1/3}\,dt$

$= \underbrace{8\left(\frac{3}{4}t^{4/3}\right)}_{\text{from part (a)}} + C$

$= 6t^{4/3} + C$ ← general solution for $R(t)$

② Now solve for C:

$0 = R(0)$

$0 = 6(0)^{4/3} + C$

$0 = C$

$\Rightarrow R(t) = 6t^{4/3}$, so solve $2 = 6t^{4/3}$

↑

particular solution for $R(t)$

$\frac{1}{3} = \frac{2}{6} = t^{4/3}$

$\left(\frac{1}{3}\right)^{3/4} = \left(t^{4/3}\right)^{3/4}$

$\left(\frac{1}{3}\right)^{3/4} = t^{\frac{4}{3}\cdot\frac{3}{4}} = t^1 = t$

$\boxed{\left(\frac{1}{3}\right)^{3/4} = t}$

Note: Since we knew the initial value $R(0) = 0$, we could have found the particular $R(t)$ in part (a). Then we could have evaluated the particular $R(12) - R(3)$.

Evaluating a particular solution and subtracting to find a change in a quantity is the same as calculating the definite integral to find the change.