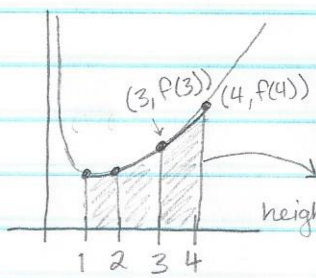
Lesson 33: Numerical Integration

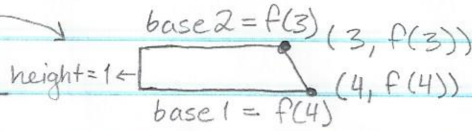
* Trapezoidal Rule – approximate the area under $f(x)$ on $[a, b]$ with n trapezoids

Ex.



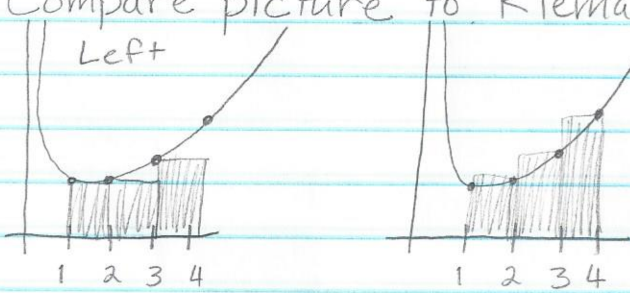
$f(x) = \frac{e^x}{x}$ on $[1, 4]$ with $n = 3$

Area of a trapezoid $= \frac{1}{2}(\text{base } 1 + \text{base } 2)\,\text{height}$



$$\text{Total Area} = \frac{f(1)+f(2)}{2}(1) + \frac{f(2)+f(3)}{2}(1) + \frac{f(3)+f(4)}{2}(1)$$
$$= \frac{1}{2}\left[f(1) + 2f(2) + 2f(3) + f(4)\right]$$
$$= \frac{1}{2}\left[\frac{e^1}{1} + 2\frac{e^2}{2} + 2\frac{e^3}{3} + \frac{e^4}{4}\right]$$
$$\approx \boxed{18.57}$$

Compare picture to Riemann Sums with $n = 3$



The trapezoidal rule gives us a better approximation than left or right Riemann Sums.
Also, note that we don't have a way to evaluate the exact value of $\int_1^4 \frac{e^x}{x}\,dx$, so all we can do is approximate the value.

* In general: height $= \boxed{\begin{aligned} \Delta x &= \frac{b-a}{n} \\ x_i &= a + i\Delta x \end{aligned}}$, $i = 0, \ldots, n$

$$\boxed{T_n = \frac{1}{2} \cdot \Delta x \cdot \left[f(x_0) + 2f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-1}) + f(x_n)\right]}$$

Ex.1 Use the TR with 4 trapezoids to approximate $\int_{-1}^{2} \ln(x+2)dx$.

$n = 4$

$\Delta x = \frac{2-(-1)}{4} = \frac{3}{4}$

$x_i = -1 + \frac{3}{4} i$

$f(x_i) = \ln(x_i + 2)$

$= \ln(-1 + \frac{3}{4}i + 2)$

$= \ln(1 + \frac{3}{4}i)$

$T_4 = \frac{1}{2}\Delta x\,(f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4))$

$= \frac{1}{2}(\frac{3}{4})(\ln(1+\frac{3}{4}\cdot 0) + 2\ln(1+\frac{3}{4}\cdot 1) + 2\ln(1+\frac{3}{4}\cdot 2) + 2\ln(1+\frac{3}{4}\cdot 3) + \ln(1+\frac{3}{4}\cdot 4))$

$= \frac{3}{8}(\ln(1) + 2\ln(\frac{7}{4}) + 2\ln(\frac{10}{4}) + 2\ln(\frac{13}{4}) + \ln(4))$

$\approx \boxed{2.51}$

Ex.2 Use the TR with 5 trapezoids to approximate $\int_{-3}^{2} x^2 dx$.

$n = 5$

$\Delta x = \frac{2-(-3)}{5} = 1$

$x_i = -3 + i$

$f(x_i) = x_i^2$

$= (-3+i)^2$

$T_5 = \frac{1}{2}\Delta x(f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + 2f(x_4) + f(x_5))$

$= \frac{1}{2}(f(-3) + 2f(-2) + 2f(-1) + 2f(0) + 2f(1) + f(2))$

$= \frac{1}{2}(9 + 2(4) + 2(1) + 2(0) + 2(1) + (4))$

$= \frac{1}{2}(9 + 8 + 2 + 0 + 2 + 4)$

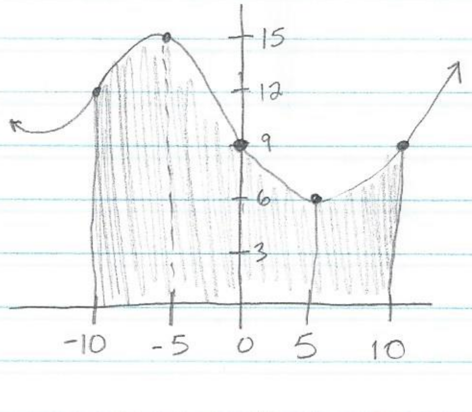
$= \frac{1}{2}(25) = \boxed{\frac{25}{2}}$

Find the exact value of $\int_{-3}^{2} x^2 dx$.

$\int_{-3}^{2} x^2 dx = \frac{1}{3}x^3\Big|_{-3}^{2} = \frac{1}{3}(2)^3 - \frac{1}{3}(-3)^3 = \frac{1}{3}(8) + \frac{1}{3}(27)$

$= \frac{8}{3} + 9 = \frac{8+27}{3} = \boxed{\frac{35}{3}}$

Ex.3 Approximate the shaded area using the TR with $n=4$.



$\Delta x = \frac{10-(-10)}{4} = 5$

$x_0 = -10,\ x_1 = -5,\ x_2 = 0,\ x_3 = 5,\ x_4 = 10$

$f(x_0) = 12,\ f(x_1) = 15,\ f(x_2) = 9,\ f(x_3) = 6,\ f(x_4) = 9$

$T_4 = \frac{1}{2}\Delta x\,(f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4))$

$= \frac{5}{2}(12 + 2(15) + 2(9) + 2(6) + 9)$

$= \frac{5}{2}(12 + 30 + 18 + 12 + 9)$

$= \frac{5}{2}(81) = \boxed{\frac{405}{2}}$