Lesson 34: Exponential Growth

✱ If the rate of change of y is proportional to y, i.e. $\frac{dy}{dt} = ky$ where k is the growth rate, then $\boxed{y(t) = Ce^{kt}}$

Check that $\frac{dy}{dt} = ky$ for $y(t) = Ce^{kt}$. → Check: $\frac{dy}{dt} = y'(t) = (Ce^{kt})' = C(e^{kt})' = C(ke^{kt}) = k(Ce^{kt}) = ky$ ✓

Note that we need initial data to solve for C.

✱ In particular, if you deposit P dollars in a savings account in which interest is compounded *continuously* at a rate r, then at time t (in years), the amount of money in the account is $\boxed{A(t) = Pe^{rt}}$

✱ Pay attention to units! (Especially for time.)

Ex. 1 Given $\frac{dy}{dt} = \underbrace{2}_{k=2}y$ and $y(2) = 100$. Find $y(4)$.

① $y(t) = Ce^{2t}$

② Use $y(2) = 100$ to solve for C.

$100 = y(2)$

$100 = Ce^{2(2)}$

$100 = Ce^{4}$

$\frac{100}{e^4} = C$

$100e^{-4} = C$

③ $y(t) = (100e^{-4})e^{2t}$

$y(t) = 100e^{2t-4}$

④ $y(4) = 100e^{2(4)-4}$

$= \boxed{100e^{4}}$

≈ 5459.8

Ex. 2 Given $\frac{dy}{dt} = \underbrace{3}_{k=3}y$ and $y(0) = 100$. Find $y(t)$.

① $y(t) = Ce^{3t}$

② $100 = y(0)$

$100 = Ce^{3(0)}$

$100 = C\cancel{e^{0}}^{\,1}$

$100 = C$

③ $\boxed{y(t) = 100e^{3t}}$

Ex. 3 A population of self-replicating nanites, $P(t)$, where t is time in days, is growing at a rate that is proportional to the population itself, and the growth rate is 0.1. The initial population is 100.

$\frac{dP}{dt} = kP$, so $P(t) = Ce^{kt}$

$k = 0.1$ $\quad P(0) = 100$

(a) What is the population after 10 days?

① $P(t) = Ce^{kt}$

$P(t) = Ce^{0.1t}$

② $100 = P(0)$

$100 = C\cancel{e^{0.1(0)}}^{1}$

$100 = C$

③ $P(t) = 100e^{0.1t}$

④ $P(10) = 100e^{(0.1)(10)}$

$= 100e^{1} \approx \boxed{272}$

(b) How long does it take for the population to double?

We want to find t so that $P(t) = 2 \cdot \underbrace{P(0)}_{\text{original population}} = 2(100) = 200.$

$200 = P(t)$

$200 = 100e^{0.1t}$

$2 = e^{0.1t}$

$\ln(2) = \ln(e^{0.1t})$

$\ln(2) = 0.1t$

$\frac{\ln(2)}{0.1} = t$

$\approx \boxed{6.93 \text{ days}}$

since $e^x = y$ if and only if $\ln(y) = x$

$\ln(x)$ and e^x are inverse functions, so $\ln(e^x) = x$

OR:

$\ln(e^{0.1t}) = 0.1t\,\cancel{\ln(e)}^{1}$ (by log rules)

$= 0.1t$

Ex. 4 The Men of Letters deposited \$10 000 in a savings account in which interest is compounded continuously. After 70 years, there is \$100 000 in the account.

$P = 10\,000$ $\quad A(70) = 100\,000$ $\quad A(t) = Pe^{rt}$

(a) What is the annual rate of interest?

$A(t) = 10\,000e^{rt}$

Use $A(70) = 100\,000$ to find r.

Solve $10\,000e^{r(70)} = 100\,000$ for r.

$e^{70r} = 10$

$\ln(e^{70r}) = \ln(10)$

$70r = \ln(10)$

$r = \frac{\ln(10)}{70} \approx 0.033$

$\approx \boxed{3.3\%}$

(b) How long did it take for the money to triple?

Solve $A(t) = 3\ (A(0) = P) = 30\ 000$ for t.

$$30\ 000 = A(t)$$
$$30\ 000 = 10\ 000\, e^{0.033t}$$
$$3 = e^{0.033t}$$
$$\ln(3) = 0.033t$$
$$\frac{\ln(3)}{0.033} = t$$

Need to use the stored value. Otherwise ≈ 33.2.

$$\boxed{33.4 \text{ years}} \approx t$$

Since $r = \frac{\ln(10)}{70}$ is the exact rate, we can use this to find the exact t.

$$30\ 000 = 10000\, e^{\left(\frac{\ln(10)}{70}\right)t}$$
$$3 = e^{\left(\frac{\ln(10)}{70}\right)t}$$
$$\ln(3) = \frac{\ln(10)}{70}\, t$$
$$70 \cdot \frac{\ln(3)}{\ln(10)} = t$$
$$\boxed{33.4} \approx t$$

Ex. 5 Herbert Aurther Runcible Cadbury deposits \$10 000 in Richie's savings account in which interest is compounded continuously. It takes 10 years for the money to double.

$P = 10\ 000$ $\qquad A(t) = Pe^{rt}$

(a) What is the annual rate of interest? $\quad A(10) = 20\ 000$

$$A(t) = 10\ 000 e^{rt}$$

Use $A(10) = 20\ 000$ to find r.

$$20\ 000 = 10\ 000\, e^{r(10)}$$
$$2 = e^{10r}$$
$$\ln(2) = 10r$$

store value $\leftarrow$ $\frac{\ln(2)}{10} = r$

$$0.069 \approx r$$
$$\boxed{6.9\%}$$

(b) How much will be in the account after 18 years?

$$A(t) = 10\ 000\, e^{0.069t}$$
$$A(18) = 10\ 000\, e^{0.069(18)}$$

with stored value

$$A(18) \approx \boxed{\$34,822.02}$$

$$A(t) = 10\ 000\, e^{\left(\frac{\ln(2)}{10}\right)t}$$
$$A(18) = 10\ 000\, e^{\left(\frac{\ln(2)}{10}\right)(18)}$$
$$\approx \$34,822.02.$$

Ex. 6 Scrooge McDuck deposits \$30 000 in a savings account for Huey, Dewey, and Louie in which interest is compounded continuously. The annual rate of interest is 4.5%.

$P = 30\,000$

$r = 0.045$

(a) How much does the account have after 10 years?

$A(t) = 30\,000\, e^{0.045t}$

$A(10) = 30\,000\, e^{(0.045)(10)}$

$A(10) \approx \boxed{\$47{,}049.37}$

(b) How long does it take for the money to triple?

Find t when $A(t) = 90\,000$.

$90000 = 30\,000\, e^{0.045t}$

$3 = e^{0.045t}$

$\ln(3) = 0.045t$

$\dfrac{\ln(3)}{0.045} = t$

$\boxed{24.4 \text{ years}} \approx t$

Ex. 7 The rate of change of a population is given by $\frac{dP}{dt} = kP$. If $P(3) = 4000$ and $P(4) = 5000$, find $P(t)$.

$P(t) = Ce^{kt}$, but we are not given C or k.

Since we have two data points, we can make a system of equations to solve for the two variables C and k.

$$\begin{cases} 4000 = P(3) = Ce^{3k} \\ 5000 = P(4) = Ce^{4k} \end{cases}$$

$$\boxed{P(t) = \left(4000e^{3\ln(4/5)}\right)\left(e^{-\ln\left(\frac{4}{5}\right)t}\right)}$$

$C \neq 0$ and $e^{kt} > 0$, so we can divide the equations.

$$\frac{4000}{5000} = \frac{\cancel{C}e^{3k}}{\cancel{C}e^{4k}}$$

$$\frac{4}{5} = \frac{e^{3k}}{e^{4k}} = e^{3k-4k} = e^{-k}$$ ← Now we can take the ln of both sides.

$\ln\left(\frac{4}{5}\right) = -k$

$\boxed{-\ln\left(\frac{4}{5}\right) = k}$

Now plug k into one of the equations to solve for C.

$4000 = Ce^{3(-\ln(\frac{4}{5}))}$

$4000 = Ce^{-3\ln(\frac{4}{5})}$ → $\dfrac{4000}{e^{-3\ln(4/5)}} = C$

$\boxed{4000e^{3\ln(4/5)} = C}$