Lesson 36: Exponential Decay

* If the rate of change of y is proportional to y, i.e. $\frac{dy}{dt} = ky$ where k is the growth rate, then $\boxed{y(t) = Ce^{kt}}$

* We need initial data to solve for C!

* If k is **negative**, we have exponential <u>decay</u>. If k is **positive**, we have exponential <u>growth</u>.

Ex. 1 Given $\frac{dy}{dt} = \underbrace{-2}y$ and $y(2) = 100$. Find $y(4)$.

$k = -2$ (negative k, so we have decay)

① $y(t) = Ce^{kt}$

$y(t) = Ce^{-2t}$

② Use $y(2) = 100$ to find C.

$100 = y(2)$

$100 = Ce^{-2(2)}$

$100 = Ce^{-4}$

$\frac{100}{e^{-4}} = C$

$100e^{4} = C$

③ $y(t) = (100e^{4})e^{-2t}$

$y(t) = 100e^{4-2t}$

④ Find $y(4)$.

$y(4) = 100e^{4-2(4)}$

$y(4) = \boxed{100e^{-4}}$

Ex. 2 Given $\frac{dy}{dt} = \underbrace{-3}y$ and $y(2) = 150$. Find $y(7)$.

$k = -3$

① $y(t) = Ce^{kt}$

$y(t) = Ce^{-3t}$

② Use $y(2) = 150$ to find C.

$150 = y(2)$

$150 = Ce^{-3(2)}$

$150 = Ce^{-6}$

$\frac{150}{e^{-6}} = C$

$150e^{6} = C$

③ $y(t) = (150e^{6})e^{-3t}$

$y(t) = 150e^{6-3t}$

④ Find $y(7)$.

$y(7) = 150e^{6-3(7)}$

$y(7) = \boxed{150e^{-15}}$

$$\frac{dP}{dt} = kP$$

Ex. 3 The Alteran population, given by $P(t)$, where t is time in years, is decreasing at a rate that is proportional to the population itself. If $P = 120000$ when $t = 3$ and $P = 30000$ when $t = 5$, what is the population when $t = 4$?

Since $\frac{dP}{dt} = kP$, $P(t) = Ce^{kt}$. We are not given C or k, so we have to use the data points $P(3) = 120000$ and $P(5) = 30000$ to find C and k.

① $\begin{cases} 120\,000 = P(3) = Ce^{3k} \\ 30\,000 = P(5) = Ce^{5k} \end{cases}$

solve for C and k.

② Divide the equations.

$$\frac{120\,000}{30\,000} = \frac{Ce^{3k}}{Ce^{5k}}$$

$$4 = \frac{e^{3k}}{e^{5k}} = e^{3k-5k}$$

($C \neq 0$ and $e^{5k} > 0$, so we are not dividing by 0)

$$4 = e^{-2k}$$

$$\ln(4) = -2k$$

store! ← $\frac{\ln(4)}{-2} = k$

$$\approx -0.693$$

③ Plug k into one of the equations and solve for C.

$$120\,000 = Ce^{3\left(\frac{\ln(4)}{-2}\right)} = Ce^{-\frac{3}{2}\ln(4)}$$

$$\frac{120\,000}{e^{-3/2 \ln(4)}} = C$$

$$120\,000\, e^{\frac{3}{2}\ln(4)} = C$$

$$120\,000\, e^{\ln(4^{3/2})} = C$$

$$120\,000\,(4^{3/2}) = C$$

$$120\,000\,(8) = C$$

$$960\,000 = C$$

④ $P(t) = 960\,000\, e^{-0.693t}$ ← use stored value!

$$P(4) = 960\,000\, e^{(-0.693)(4)} = \boxed{60\,000}$$

Ex. 4 Naquadria decays to Naquadah with a half-life of approximately 5400 years. There are 30g of Naquadria now.

(a) How much of it remains after 1000 years?

This means that after 5400 years, only half of the amount remains.

In general, $\boxed{k = -\frac{\ln(2)}{t_{1/2}}}$.

We have $y(0) = 30$ g, so $y(5400) = 15$ g.

① $y(t) = Ce^{kt}$

② $y(0) = 30$

$$Ce^{(0)k} = 30$$

$$C = 30$$

③ $y(t) = 30e^{kt}$

$$y(5400) = 15$$

$$30e^{5400k} = 15$$

$$e^{5400k} = \frac{1}{2}$$

$$5400k = \ln\left(\frac{1}{2}\right)$$

$$k = \frac{\ln\left(\frac{1}{2}\right)}{5400} \approx -0.000128$$

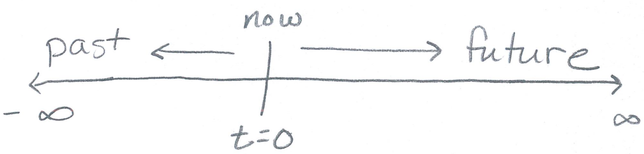
↑ store!

④ $y(t) = 30e^{kt}$ → using stored value for k.

$$y(1000) = 30e^{k(1000)}$$

$$y(1000) = 30e^{-0.128}$$

$$\approx \boxed{26.4 \text{ g}}$$



(b) How much was there 100 years ago?

Since now, $t=0$, 100 years ago, $t=-100$.

$y(-100) = 30e^{-100k}$ → use stored value!

$y(-100) = 30e^{0.0128}$

$\approx \boxed{30.4\text{ g}}$

Ex. 5 Naquadria decays to Naquadah with a half-life of approximately 5400 years. After 50 years, there are 100g of Naquadria. ↳ We can use the previous k value because we have the same elements with the same half-life. $k \approx -0.000128$

(a) What was the initial quantity?

We know $y(50) = 100$. We want to find $y(0)$.

$y(t) = Ce^{-0.000128t}$

$y(50) = 100$

$Ce^{(-0.000128)(50)} = 100$

$C = \dfrac{100}{e^{(-0.000128)(50)}}$

$C = 100e^{(0.000128)(50)}$

store! ← $C \approx 100.644$ g

$y(t) = 100.644e^{-0.000128t}$

$y(0) = 100.644\cancel{e^{0}}^{\,1}$

$y(0) = \boxed{100.644} = C$

(b) How much remains after 12000 years?

$y(t) = 100.644e^{-0.000128t}$

$y(12000) = 100.644e^{(-0.000128)(12000)}$

$\approx \boxed{21.569}$

Ex. 6 Naquadria decays to Naquadah with a half-life of approximately 5400 years. → Can still use same k value.

(a) What percent of a given amount remains after 7000 years?

Assume there is initially 100% Naquadria.
Then $y(0) = 100\%$, so $y(t)$ gives the % of Naquadria that remains after t years.

① $y(t) = Ce^{kt}$

$$y(0) = Ce^{0} = 100$$
$$C = 100$$

② $y(t) = 100e^{kt}$ using stored k value.

$$y(7000) = 100e^{k(7000)}$$
$$\approx \boxed{40.7\%}$$

(b) A deposit on Langara contains 20% Naquadria and 80% Naquadah. How long ago did Thanos convert the Naquadah to Naquadria?

There is 20% Naquadria remaining of an initial 100% Naquadria.
Since $y(t)$ gives us the % of Naquadria remaining after t years, we need to solve $20\% = y(t)$ for t.

stored ≈ -0.000128

$$20 = 100e^{kt}$$
$$\frac{20}{100} = e^{kt}$$
$$\frac{1}{5} = e^{kt}$$
$$\ln\left(\frac{1}{5}\right) = kt$$
$$\frac{\ln\left(\frac{1}{5}\right)}{k} = t$$
$$\approx \boxed{12538 \text{ years}}$$ (ago)