Lesson 4: Finding Limits Analytically

*Non-Piecewise Functions

For non-piecewise functions, we can evaluate the limit

$$\lim_{x \to c} f(x)$$

analytically by finding $f(c)$. Once we evaluate, we will run into 3 potential cases.

* Case 1: If $f(c) =$ a *finite* number, then we can conclude

$$\lim_{x \to c} f(x) = f(c)$$

Ex.1 Find

$$\lim_{x \to 2} (x^3 + 3x^2 - 7)$$

For this problem, we can find the limit by evaluating $x^3 + 3x^2 - 7$ for $x = 2$. We get $2^3 + 3(2)^2 - 7 = 8 + 12 - 7 = 13$. That means

$$\lim_{x \to 2} (x^3 + 3x^2 - 7) = 13$$

* Case 2: If $f(c) = \frac{\text{nonzero number}}{0}$, then

$$\lim_{x \to c} f(x) = -\infty, \infty \text{ or DNE}$$

We can determine which of those is the limit by looking at the one-sided limits. If the left and right sided limits are both ∞, then the limit is ∞. If the left and right sided limits are both $-\infty$, then the limit is $-\infty$. If the one-sided limits do not match, then the limit does not exist.

Ex.2 Find

$$\lim_{x \to 4} \frac{1}{x-4}$$

First of all, we see that $\frac{1}{4-4} = \frac{1}{0}$, so the function $\frac{1}{x-4}$ is undefined at $x = 4$. However, when we divide by really, really small numbers as the denominator goes to 0, the fraction gets really, really big, so we know the magnitude of the one-sided limits will be going to infinity. We need to figure out if the signs of the one-sided limits match.

For the left limit, we are looking at $x < 4$, so $x - 4 < 0$ meaning it's negative. Then $\lim_{x \to 4^-} \frac{1}{x-4} = \frac{1}{\text{negative number} \to 0} = -\infty$.

For the right limit, we are looking at $x > 4$, so $x - 4 > 0$ meaning it's positive. Then $\lim_{x \to 4^+} \frac{1}{x-4} = \frac{1}{\text{positive number} \to 0} = \infty$.

Since the one-sided limits do not match, $\lim_{x \to 4} \frac{1}{x-4}$ does not exist.

Ex.3 Find

$$\lim_{x \to -1} \frac{1}{(1+x)^2}$$

Again, we see that $\frac{1}{(1+(-1))^2} = \frac{1}{0}$, so the function $\frac{1}{(1+x)^2}$ is undefined at $x = -1$ and the *magnitude* of the one-sided limits will be going to infinity, but we need to figure out if the signs of the one-sided limits match.

For the left limit, we are looking at $x < -1$, so $x + 1 < 0$. However, the denominator is squared, so we actually care about the sign of $(x+1)^2 = (1+x)^2$ which is always positive. Then $\lim_{x \to -1^-} \frac{1}{(1+x)^2} = \frac{1}{\text{positive number} \to 0} = \infty$.

For the right limit, we are looking at $x > -1$, so $x + 1 > 0$ meaning it's positive and is again squared. Then $\lim_{x \to -1^+} \frac{1}{(1+x)^2} = \frac{1}{\text{positive number} \to 0} = \infty$.

Since the one-sided limits both equal ∞, $\lim_{x \to -1} \frac{1}{(1+x)^2} = \infty$.

As a general rule, if you're finding $\lim_{x \to c} \frac{d}{(x-c)^n}$, the limit exists if n is *even* because the denominator will always be positive. The sign of the limit depends on the sign of d or whatever is in the numerator. If n is *odd*, then the limit does not exist because the sign in the denominator will change.

* Case 3: If $f(c) = \frac{0}{0}$, then we need to simplify or rationalize the function and evaluate again. Once we do that, we should get Case 1 or Case 2. In general, it's a good idea to simplify a function before finding a limit.

Note: In this course, we will not be learning L'Hôpital's Rule. If you have learned it in a previous course, you may use it on homework, quizzes, or exams. Make sure you use it correctly if you chose to use it. You will lose points on quizzes for using it incorrectly.

Ex.4 For $f(x) = \frac{x^2-3x}{x^2+5x}$, find the following limits.

$$\lim_{x\to 0} f(x), \qquad \lim_{x\to 3} f(x), \qquad \lim_{x\to -5} f(x)$$

For the limit as x approaches 0, $f(0) = \frac{0}{0}$ which is Case 3, so we need to simplify the function. $f(x) = \frac{x^2-3x}{x^2+5x} = \frac{x(x-3)}{x(x+5)} = \frac{x-3}{x+5}$. Now, we have

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{x-3}{x+5} = \frac{-3}{5}$$

Now, we look at the limit as x approaches 3. We can continue using the simplified function to find the limits. $f(3) = \frac{3-3}{3+5} = \frac{0}{8} = 0$. Note that this is Case 1 because 0 is a finite number.

$$\lim_{x\to 3} f(x) = \lim_{x\to 3} \frac{x-3}{x+5} = 0$$

Finally, we look at the limit as x approaches -5. We have $f(-5) = \frac{-5-3}{-5+5} = \frac{-8}{0}$, so this is Case 2. We need to look at the one-sided limits to determine if the limit exists and what it is. For the left limit, we have $x < -5$, so $x + 5 < 0$ which means

$$\lim_{x\to -5^-} \frac{x-3}{x+5} = \frac{-5-3}{\text{negative number} \to 0} = \infty$$

We get positive infinity because the numerator and denominator are both negative for this problem.

For the right limit, we have $x > -5$, so $x + 5 > 0$ which means

$$\lim_{x\to -5^+} \frac{x-3}{x+5} = \frac{-5-3}{\text{positive number} \to 0} = -\infty$$

Since the limits do not match, $\lim_{x\to -5} f(x)$ does not exist.

*Piecewise Functions

For piecewise functions, if we're looking where the 2 pieces of a function meet, then we MUST look at the left and right limits.

<u>Ex.5</u> For $f(x) = \begin{cases} \sin(x) & x \leq \pi \\ 1 + \cos(x) & x > \pi \end{cases}$, find $\lim_{x\to\pi} f(x)$.

Looking at the left limit, we are considering $x < \pi$, so we use $\sin(x)$.

$$\lim_{x\to\pi^-} f(x) = \lim_{x\to\pi^-} \sin(x) = \sin(\pi) = 0$$

Looking at the right limit, we are considering $x > \pi$, so we use $1 + \cos(x)$.

$$\lim_{x\to\pi^+} f(x) = \lim_{x\to\pi^+} (1 + \cos(x)) = 1 + \cos(\pi) = 1 + (-1) = 0$$

Since the left and right limits are both 0,

$$\lim_{x\to\pi} f(x) = 0$$

<u>Ex.6</u> For $f(x) = \begin{cases} \sqrt[3]{x} & x \leq -8 \\ \frac{1}{4}\ln\left(\frac{1}{e^x}\right) & x > -8 \end{cases}$, find $\lim_{x\to-8} f(x)$.

For the left limit, $x < -8$, so we use the function $\sqrt[3]{x}$ and get

$$\lim_{x\to-8^-} f(x) = \lim_{x\to-8^-} \sqrt[3]{x} = \sqrt[3]{-8} = -2$$

For the right limit, $x > -8$, so we use the function $\frac{1}{4}\ln\left(\frac{1}{e^x}\right)$ and get

$$\lim_{x\to-8^+} f(x) = \lim_{x\to-8^+} \frac{1}{4}\ln\left(\frac{1}{e^x}\right) = \frac{1}{4}\ln\left(\frac{1}{e^{-8}}\right) = \frac{1}{4}\ln(e^8) = \frac{1}{4}\cdot(8) = 2$$

Since the left and right limits are not equal, $\lim_{x\to-8} f(x)$ does not exist.