Lesson 5: Continuity

Def $f(x)$ is continuous at $x = c$ if

① $f(c)$ is defined

② $\lim_{x \to c} f(x)$ exists

③ $\lim_{x \to c} f(x) = f(c)$

Generally, if we can trace the function without lifting our pencil/finger.

* Where should we look for discontinuities?

- Non-piecewise functions
  Before simplifying, set the denominator equal to 0. That is where $f(x)$ must be undefined, so $f(x)$ is discontinuous there.
- Piecewise functions
  Check where the pieces meet. The function might be discontinuous there.

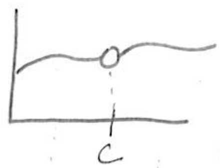
* Types of Discontinuities

Def. $f(x)$ has a hole at $x=c$ if

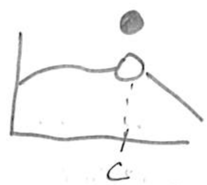
① $\lim_{x \to c} f(x)$ exists and is finite

and

② $\lim_{x \to c} f(x) \neq f(c)$



f(c) undefined,
hole at $x=c$



f(c) defined,
hole at $x=c$

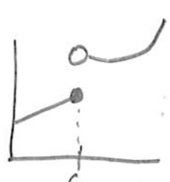
Def. $f(x)$ has a jump at $x=c$ if

① $\lim_{x \to c} f(x)$ DNE

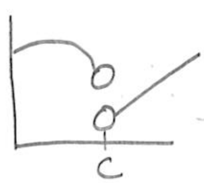
and

② $\lim_{x \to c^-} f(x)$ and $\lim_{x \to c^+} f(x)$ are finite

③ $\lim_{x \to c} f(x) \neq f(c)$



f(c) defined,
jump at $x=c$



f(c) undefined,
jump at $x=c$

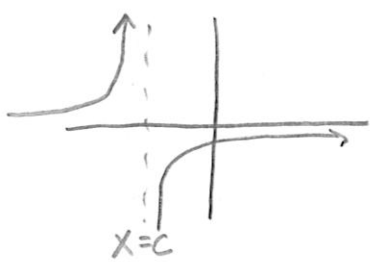
Note: Jumps only happen for piecewise functions.

Def. $f(x)$ has a <u>vertical asymptote</u> at $x=c$ if

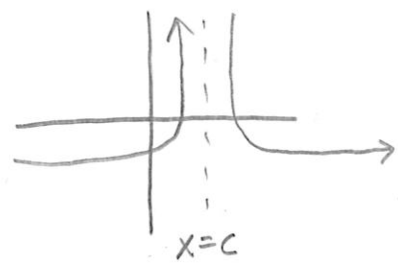
① $f(c)$ is undefined

<u>and</u>

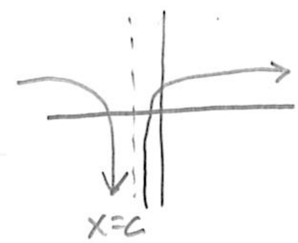
② $\lim_{x \to c^-} f(x)$ and $\lim_{x \to c^+} f(x)$ are ∞ or $-\infty$



$\lim_{x \to c} f(x)$ DNE,
VA at $x = c$

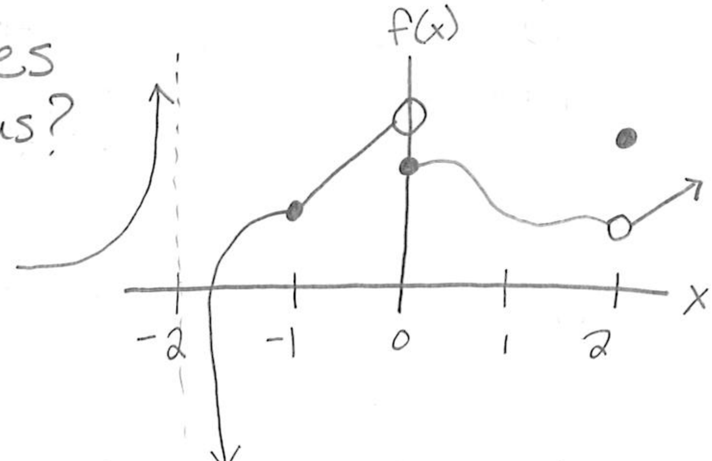


$\lim_{x \to c} f(x) = \infty$,
VA at $x=c$



$\lim_{x \to c} f(x) = -\infty$,
VA at $x=c$

<u>Ex. 1</u> For what x-values is $f(x)$ discontinuous? Classify each discontinuity.



<u>Note</u>: The values of $f(x)$ do <u>not</u> matter.

VA @ $x = -2$

Jump @ $x = 0$

Hole @ $x = 2$

Find where $f(x)$ is discontinuous (if anywhere), and classify the discontinuities.

Ex. 2 $\quad f(x) = \dfrac{x^2+3x-4}{x^2+4x-5}$

First, $\quad x^2+4x-5=0$

$(x-1)(x+5)=0$

$x=1,\ x=-5$

so $f(x)$ is discontinuous at $x=1$ and $x=-5$. Now to classify.

Simplify: $f(x) = \dfrac{\cancel{(x-1)}(x+4)}{\cancel{(x-1)}(x+5)} = \dfrac{x+4}{x+5}$.

$x=1$: $\displaystyle\lim_{x\to 1} \frac{x+4}{x+5} = \frac{5}{6}$, so the limit exists and is finite, but $f(1)$ is undefined. Therefore $\boxed{\text{Hole @ } x=1}$

$x=-5$: $\displaystyle\lim_{x\to -5} \frac{x+4}{x+5} = \frac{-1}{0}$, so the one-sided limits will be infinite, which means $\boxed{\text{VA @ } x=-5}$

LON-CAPA: $\boxed{-5}$ $\boxed{1}$

$\boxed{\text{VA}}$ $\boxed{\text{Hole}}$

Note: In general, if we can cancel out a factor from the denominator, we have a hole. If the factor remains in the denominator after simplifying, we have a VA.

Ex.3 $f(x) = \begin{cases} x-1 & , x \neq 2 \\ -2 & , x = 2 \end{cases}$

Note that $x \neq 2$ is the same as $x > 2$ and $x < 2$, so

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} (x-1) = 2-1 = 1$$

But, $f(2) = -2$.

Therefore, $\boxed{\text{hole at } x = 2}$

LON-CAPA! $\boxed{2}$ $\boxed{\text{DNE}}$
$\boxed{\text{Hole}}$ $\boxed{\text{DNE}}$

Ex.4

$$f(x) = \begin{cases} x^2+1 & , x \leq 0 \\ \cos(x) & , 0 < x < \pi \\ \frac{x}{\pi} - 2 & , x \geq \pi \end{cases}$$

$f(x)$ is defined at $x=0$ and $x=\pi$, but we need to check the limits.

$x=0$:

$$\lim_{\substack{x \to 0^- \\ x<0}} (x^2+1) = 0^2+1 = 1$$

$$\lim_{\substack{x \to 0^+ \\ x>0}} \cos(x) = \cos(0) = 1$$

$\Rightarrow \lim_{x \to 0} f(x) = 1$ and $f(0) = 1$, so continuous at $x=0$

$x=\pi$:

$$\lim_{\substack{x \to \pi^- \\ x<\pi}} \cos(x) = \cos(\pi) = -1$$

$$\lim_{\substack{x \to \pi^+ \\ x>\pi}} \left(\frac{x}{\pi} - 2\right) = \left(\frac{\pi}{\pi} - 2\right) = 1-2 = -1$$

$\Rightarrow \lim_{x \to \pi} f(x) = -1$ and $f(\pi) = -1$, so continuous at $x=\pi$

LON-CAPA! $\boxed{\text{DNE}}$ $\boxed{\text{DNE}}$
$\boxed{\text{DNE}}$ $\boxed{\text{DNE}}$

Ex.5 $f(x) = \begin{cases} e^{-2x}, & x \le 0 \\ \ln(x+1), & x > 0 \end{cases}$

$$\left.\begin{array}{l} \lim\limits_{x\to 0^-} (e^{-2x}) = e^0 = 1 \\ \lim\limits_{x\to 0^+} (\ln(x+1)) = \ln(0+1) = \ln(1) = 0 \end{array}\right) \neq$$

$\lim\limits_{x\to 0} f(x)$ DNE but finite one-sided limits. Therefore,

$\boxed{\text{Jump @ } x=0}$

Ex.6 $f(x) = \dfrac{x^3 + 2x^2 + x}{x^3 - x^2}$

First, $x^3 - x^2 = 0$

$x(x^2-1) = 0$

$x(x-1)(x+1) = 0$

so discontinuous at $x=0$, $x=1$, $x=-1$

Now, simplify:

$$f(x) = \frac{x(x^2+2x+1)}{x(x-1)(x+1)} = \frac{\cancel{x}(x+1)\cancel{(x+1)}}{\cancel{x}(x-1)\cancel{(x+1)}} = \frac{x+1}{x-1}$$

$$\boxed{\begin{array}{l} \text{Hole @ } x=-1 \\ \text{Hole @ } x=0 \\ \text{VA @ } x=1 \end{array}}$$