Lesson 6: The Derivative

Def. A **difference quotient** for a function has the form

$$\frac{f(x+h) - f(x)}{(x+h) - x}$$
$$\frac{f(x+\Delta x) - f(x)}{(x+\Delta x) - x}$$
$$\frac{f(a+h) - f(a)}{(a+h) - a}$$

Notice that a difference quotient always has the form of "change in y" divided by "change in x" which should make us think of the slope of a line. In fact, that's exactly what it is, the slope of a line passing through 2 points.

<u>Def.</u> The <u>derivative</u> of a function is the slope of the tangent line to the function.



Looking at the top graph, the dashed line is the tangent line to f(x) at the point. If we want to find the slope of this line, we need 2 points on the line, but we only have the 1 point. However, we can estimate the tangent line by looking at a point *close* to the point.

Looking at the bottom graph, we have picked such a point. The slope of this line has the form of the difference quotient defined above.

$$slope = \frac{f(x+h) - f(x)}{(x+h) - x} = \frac{f(x+h) - f(x)}{h}$$

If h goes to 0, then we're moving towards the tangent line at the actual point where we want to find the slope. This leads to another definition of the derivative. **<u>Def.</u>** The <u>derivative</u> of f(x) is given by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

When we use the definition of the derivative to find a derivative, we are using the "limit process" to find the derivative.

There are different notations for the derivative. Most commonly, we see f'(x), $\frac{dy}{dx}$, y', and $\frac{d}{dx}f(x)$.

<u>Ex.1</u> Find the derivative of f(x) = 3 - x. We need to find

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

We have...

$$f(x+h) = 3 - (x+h) = 3 - x - h$$

$$f(x+h) - f(x) = (3 - x - h) - (3 - x) = 3 - x - h - 3 + x = -h$$

$$\frac{f(x+h) - f(x)}{h} = \frac{-h}{h} = -1$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} (-1) = -1$$

Since we're taking the limit as h goes to 0 of -1, and -1 doesn't depend on h, the limit is just -1. Be careful with parentheses! You need to distribute all negatives!

<u>Ex.2</u> Find the derivative of $f(x) = \frac{1}{x-1}$. We're going to go through the limit process of finding the derivative.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{(x+h)-1} - \frac{1}{x-1}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{x-1}{(x+h-1)(x-1)} - \frac{x+h-1}{(x-1)(x+h-1)}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{(x-1) - (x+h-1)}{(x+h-1)(x-1)}}{h}$$

$$= \lim_{h \to 0} \frac{x-1 - x - h + 1}{(x+h-1)(x-1)} \cdot \frac{1}{h}$$

$$= \lim_{h \to 0} \frac{-h}{(x+h-1)(x-1)} \cdot \frac{1}{h}$$

$$= \lim_{h \to 0} \frac{-1}{(x+h-1)(x-1)}$$

$$= \frac{-1}{(x-1)^2}$$

Looking at the simplifications above, we first find a common denominator for the *numerator* to combine the 2 fractions. Then we simplify until we get rid of the $\frac{1}{h}$. Finally, we can take the limit with $h \to 0$ to get rid of h and get the derivative. **Ex.3** Find the equation of the tangent line to the graph of $f(x) = 2 - \sqrt{x}$ at x = 1. Finding the equation of a tangent line to f(x) at x = c is a very important concept that will appear throughout this course. I break the process into 3 steps. Note that the equation of a tangent *line* will ALWAYS have the form y = mx + b. Step 1: Find the slope of the tangent line to f(x) at x = c, i.e. m = f'(c). Step 2: Find a point where the tangent line touches the graph at x = c, i.e. the point (c, f(c)).

Step 3: Use the slope and point to find the equation of the tangent line. Now let's work through this example.

Step 1: Find m = f'(1).

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h \to 0} \frac{(2 - \sqrt{x+h}) - (2 - \sqrt{x})}{h}$$
$$= \lim_{h \to 0} \frac{2 - \sqrt{x+h} - 2 + \sqrt{x}}{h}$$
$$= \lim_{h \to 0} \frac{-\sqrt{x+h} + \sqrt{x}}{h}$$

At this point, we still need to cancel the h from the denominator so that the limit is something we can evaluate. Unfortunately, the h in the numerator is trapped inside a square root. To liberate h from this annoying prison, we need to rationalize the expression in the numerator. Remember from pre-calculus that rationalizing something uses the difference of 2 squares: $a^2 - b^2 = (a - b)(a + b)$. We need to recognize if we have (a - b) or (a + b) then multiply by the conjugate (the factor with the opposite sign). Then we have the difference of 2 squares and we can square the terms, so the variables or numbers are no longer trapped under the square root. In this instance, we have (a + b) with $a = \sqrt{x + h}$ and $b = \sqrt{x}$. This means we are going to multiply and divide the fraction by the conjugate which is $(a - b) = -\sqrt{x + h} - \sqrt{x}$. (We could look at $\sqrt{x} - \sqrt{x + h}$ and use different definitions of a and b, but let's continue with the order as written above.)

$$f'(x) = \lim_{h \to 0} \frac{-\sqrt{x+h} + \sqrt{x}}{h} \cdot \frac{-\sqrt{x+h} - \sqrt{x}}{-\sqrt{x+h} - \sqrt{x}}$$
$$= \lim_{h \to 0} \frac{(-\sqrt{x+h})^2 - (\sqrt{x})^2}{h(-\sqrt{x+h} - \sqrt{x})}$$
$$= \lim_{h \to 0} \frac{(x+h) - (x)}{h(-\sqrt{x+h} - \sqrt{x})}$$
$$= \lim_{h \to 0} \frac{h}{h(-\sqrt{x+h} - \sqrt{x})}$$
$$= \lim_{h \to 0} \frac{1}{(-\sqrt{x+h} - \sqrt{x})}$$
$$= \frac{1}{-2\sqrt{x}}$$

Now that we have $f'(x) = -\frac{1}{2\sqrt{x}}$, we can find the slope $m = f'(1) = -\frac{1}{2}$.

Step 2: Find the point where the tangent line and function touch, i.e. (1, f(1)). $f(1) = 2 - \sqrt{1} = 1$

Step 3: Find the equation of the tangent line y = mx + b.

We have 2 methods that we usually use to find the equation of line. We can use point-slope form or we can find the y-intercept.

y-intercept Method: Plug the slope $m = -\frac{1}{2}$ and point (x, y) = (1, 1) into y = mx + b and solve for b.

$$y = mx + b$$

$$1 = \left(-\frac{1}{2}\right)(1) + b$$

$$1 = -\frac{1}{2} + b$$

$$1 + \frac{1}{2} = b$$

$$\frac{3}{2} = b$$

Now we plug m and b into y = mx + b and we have the tangent line $y = -\frac{1}{2}x + \frac{3}{2}$. *Point-slope method* Start with $y - y_1 = m(x - x_1)$. Plug in $m = -\frac{1}{2}$ and $(x_1, y_1) = (1, 1)$ then solve for y.

$$y - y_1 = m(x - x_1)$$

$$y - 1 = -\frac{1}{2}(x - 1)$$

$$y - 1 = -\frac{1}{2}x + \frac{1}{2}$$

$$y = -\frac{1}{2}x + \frac{1}{2} + 1$$

$$y = -\frac{1}{2}x + \frac{3}{2}$$

You can see that both methods give the tangent line $y = -\frac{1}{2}x + \frac{3}{2}$.

Here are a few examples to try on your own:

<u>Ex.4</u> Find the derivative of $f(x) = \sqrt{2-x}$. **Answer:** -1

$$f'(x) = \frac{-1}{2\sqrt{2-x}}$$

<u>Ex.5</u> Find the equation of the tangent line to the graph of $f(x) = x^2 + 4$ at x = 0. **Answer:** Slope is m = f'(0) = 0. Equation of the tangent line is y = 4.