Lesson 7: Basics of Differentiation; $\sin x$, $\cos x$, e^x

* Constant Rule: $\frac{d}{dx}(c) = 0$

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{c - c}{\Delta x} = \lim_{\Delta x \to 0} \frac{0}{\Delta x} = \lim_{\Delta x \to 0} 0 = 0$$

↳ 0 divided by any # is 0, so $\frac{0}{\Delta x} = 0$.

* Power Rule: $\frac{d}{dx}(x^n) = n x^{n-1}$

Note: n can be a fraction, negative number, etc.

Easy to see for $n = 1, 2,$ and 3.

$$f(x) = x^1 \Rightarrow f'(x) = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^1 - (x)^1}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1 = 1 \cdot x^0$$

$$\begin{aligned} f(x) = x^2 \Rightarrow f'(x) &= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - x^2}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 - x^2}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{2x\Delta x + (\Delta x)^2}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{\Delta x(2x + \Delta x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} (2x + \Delta x) \\ &= 2x = 2 \cdot x^1 \end{aligned}$$

$f(x) = x^3 \Rightarrow$ Try on your own (worked out on LON-CAPA)

* Constant Multiple Rule: $\frac{d}{dx}(cf(x)) = c\frac{d}{dx}(f(x)) = cf'(x)$

* Sum Rule: $\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$

* Difference Rule: $\frac{d}{dx}(f(x) - g(x)) = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$

Note: Since we define derivatives in terms of limits, these rules follow from the limit properties.

Ex.1 If $f(x) = x^5 - 3x^2 + x^{1.5} - \frac{1}{\sqrt{x}}$, find $f'(x)$.

Note! $\frac{1}{\sqrt{x}} = x^{-1/2}$

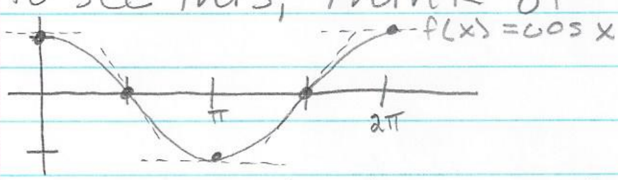
$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(x^5 - 3x^2 + x^{1.5} - x^{-1/2}\right) \\ &= \frac{d}{dx}(x^5) - 3\frac{d}{dx}(x^2) + \frac{d}{dx}(x^{1.5}) - \frac{d}{dx}(x^{-1/2}) \\ &= 5x^4 - 3\cdot 2x + 1.5x^{1.5-1} - \left(-\tfrac{1}{2}\right)x^{-\frac{1}{2}-1} \\ &= 5x^4 - 6x + 1.5x^{0.5} + \tfrac{1}{2}x^{-3/2} \end{aligned}$$

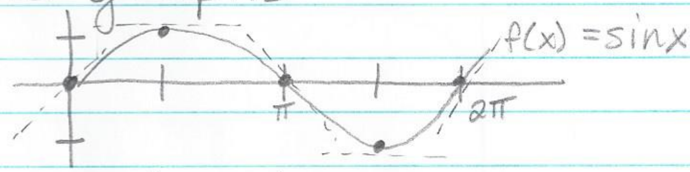
Ex.2 If $f(x) = x^{4e} - \sqrt[5]{x^{-4}}$, find $f'(x)$.

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(x^{4e} - \sqrt[5]{x^{-4}}\right) \\ &= \frac{d}{dx}\left(x^{4e} - x^{-4/5}\right) \\ &= \frac{d}{dx}(x^{4e}) - \frac{d}{dx}(x^{-4/5}) \\ &= 4e\,x^{4e-1} - \left(-\tfrac{4}{5}\right)x^{-4/5-1} \\ &= 4e\,x^{4e-1} + \tfrac{4}{5}x^{-9/5} \end{aligned}$$

* $\frac{d}{dx}(\cos x) = -\sin x$ $\qquad$ $\frac{d}{dx}(\sin x) = \cos x$

To see this, think of the graphs:





slope: 0 $\;$ -1 $\;$ 0 $\;$ 1 $\;$ 0 $\qquad$ Slope: 1 $\;$ 0 $\;$ -1 $\;$ 0 $\;$ 1

Draw some tangent lines, and you can see how sin x and cos x are related.

Ex.3 If $f(x) = \cos x - 2\sin x$, find $f'\left(-\frac{\pi}{2}\right)$.

$$\begin{aligned} f'(x) &= \frac{d}{dx}(\cos x - 2\sin x) \\ &= \frac{d}{dx}\cos x - 2\frac{d}{dx}\sin x \\ &= -\sin x - 2\cos x \end{aligned}$$

$$f'\left(-\tfrac{\pi}{2}\right) = -\sin\left(-\tfrac{\pi}{2}\right) - 2\cos\left(-\tfrac{\pi}{2}\right) = \boxed{1}$$

* $\frac{d}{dx}(e^x) = e^x$ This is harder to see, but from numerical approximation, we can show it.

Ex.4 If $f(x) = \frac{1}{x^7} - \cos x + 7e^x - 1$, find $f'(x)$.

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(\frac{1}{x^7} - \cos x + 7e^x - 1\right) \\ &= \frac{d}{dx}(x^{-7}) - \frac{d}{dx}\cos x + 7\frac{d}{dx}e^x - \frac{d}{dx}(1) \\ &= -7x^{-6} - (-\sin x) + 7e^x + 0 \\ &= -7x^{-6} + \sin x + 7e^x \end{aligned}$$

Ex.5 Find the equation of the tangent line to the graph of $y = x^{3/2} + \cos x - 2e^x$ at $x = 0$.

Slope of tan line: $$\begin{aligned} y' &= \frac{d}{dx}(x^{3/2} + \cos x - 2e^x) \\ &= \frac{d}{dx}x^{3/2} + \frac{d}{dx}\cos x - 2\frac{d}{dx}e^x \\ &= \frac{3}{2}x^{1/2} - \sin x - 2e^x \end{aligned}$$
$$y'|_{x=0} = \frac{3}{2}(0)^{1/2} - \sin(0) - 2e^0 = -2$$

Equation for tan line: $(0, y|_{x=0}) = (0, -1)$ is on tan line

$$\begin{aligned} y - (-1) &= -2(x - 0) \\ y + 1 &= -2x \\ \boxed{y = -2x - 1} \end{aligned}$$

Ex.6 Differentiate $y = e^x - 2\sin x + \frac{1}{x^2} - \frac{1}{\sqrt{x}}$.

$$\begin{aligned} y' &= (e^x)' - 2(\sin x)' + (x^{-2})' - (x^{-1/2})' \\ &= e^x - 2\cos x - 2x^{-3} + \frac{1}{2}x^{-3/2} \end{aligned}$$

Ex.7 Differentiate $y = \sin x - \cos x + 7$.

$$\begin{aligned} y' &= (\sin x)' - (\cos x)' + (7)' \\ &= \cos x + \sin x \end{aligned}$$

Ex.8 Find $f'(0)$ when $f(x) = 2e^x - \sin x$.

$f'(x) = 2(e^x)' - (\sin x)' = 2e^x - \cos x \Rightarrow f'(0) = 2 - 1 = \boxed{1}$