Lesson 8: Instantaneous Rates of Change

Recall: $\frac{d}{dx}(c) = 0$

$\frac{d}{dx}(x^n) = nx^{n-1}$

$(cf(x))' = cf'(x)$

$(f(x)+g(x))' = f'(x)+g'(x)$

$(f(x)-g(x))' = f'(x)-g'(x)$

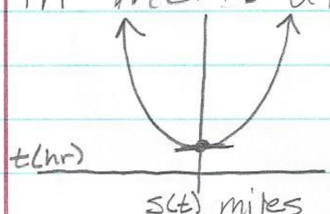
$\frac{d}{dx}(\cos x) = -\sin x$

$\frac{d}{dx}(\sin x) = \cos x$

$\frac{d}{dx}(e^x) = e^x$

* **Rate of change**

Say $s(t) = t^2+1$ is the position of a car in meters after t seconds



- To find the rate of change or velocity during the first hour, we would use $v = \frac{s(1)-s(0)}{1-0} = \frac{\Delta s}{\Delta t}$ $\frac{\text{meters}}{\text{second}}$

- If we want to find the instantaneous rate of change, we need $\Delta t \to 0$, so we need the derivative.

* **Velocity**: $v(t) = \frac{ds}{dt}$ $\left(\frac{m}{s}\right)$

What if we take the derivative of velocity?
Think about units: $\frac{dv}{dt}$ $\frac{(m/s)}{s} = \frac{m}{s^2}$

* **Acceleration**: $a(t) = \frac{dv}{dt}$ $\left(\frac{m/s}{s} = \frac{m}{s^2}\right)$

* In general, the instantaneous rate of change at $t=c$ for a function $f(t)$ is given by $f'(c)$ (or $\frac{df}{dt}\big|_{t=c}$).

* Pay attention to units!

Ex.1 The position of a womp rat at time t (in min) is given by $f(t) = t^4 - \sqrt{t^5} + \cos t + 7$. Find its acceleration function $a(t)$.

First, find velocity function:

$$
\begin{aligned}
v(t) &= f'(t) \\
&= (t^4 - t^{5/2} + \cos t + 7)' \\
&= (t^4)' - (t^{5/2})' + (\cos t)' + (7)' \\
&= 4t^3 - \tfrac{5}{2}t^{3/2} - \sin t
\end{aligned}
$$

Now, take derivative of $v(t)$ to find $a(t)$.

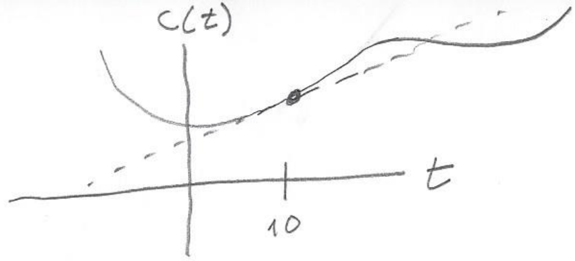
$$
\begin{aligned}
a(t) &= v'(t) \\
&= (4t^3 - \tfrac{5}{2}t^{3/2} - \sin t)' \\
&= 4(t^3)' - \tfrac{5}{2}(t^{3/2})' - (\sin t)' \\
&= 4 \cdot 3t^2 - \tfrac{5}{2} \cdot \tfrac{3}{2} t^{1/2} - \cos t \\
&= \boxed{12t^2 - \tfrac{15}{4}\sqrt{t} - \cos t}
\end{aligned}
$$

* Ex.6 The government estimates the amount of money it spends on the Stargate program is given by $C(t) = -0.5t^2 + 12t + 11$ where $C(t)$ is the cost in thousands of dollars, t is the time the Stargate is open in minutes, and $0 \le t \le 38$. When the Stargate is open for 10 minutes, what is the rate of change of the cost?

$$
\begin{aligned}
C'(t) &= -0.5(t^2)' + 12(t)' + (11)' \\
&= -0.5(2t) + 12 \cdot 1 + 0 \\
&= -t + 12
\end{aligned}
$$

$$C'(10) = -10 + 12 = 2$$

Note: How can we interpret this? The rate of change of cost is >0. Unit of $C'(t)$ is $\frac{\text{thousands of \$}}{\text{min}}$, so after the



Stargate is open for 10 minutes, the cost is increasing by \$2,000/min.
This rate helps us estimate cost for future times.

Ex. 2 The population of iratus bugs on a planet since the year 2000 can be modeled as $P(t) = 10(5t^2 - 2t + 100)$, where t is months since January 2000. In which year is the population increasing at the rate of 1280 bugs per month?

Note: $P(t) = 10(5t^2 - 2t + 100) = 50t^2 - 20t + 1000$

$$P'(t) = 50(t^2)' - 20(t)' + (1000)'$$
$$= 50 \cdot 2t - 20 \cdot 1 + 0$$
$$= 100t - 20$$

Want $P'(t) = 1280 = 100t - 20 \Rightarrow t = 13$,
so 13 months after January 2000, the rate is increasing by 1280 bugs/month
$\Rightarrow$ year $\boxed{2001}$.

Ex. 4 The volume of a sphere is given by $V(r) = \frac{4}{3}\pi r^3$. What is the rate of change of the volume with respect to the radius r when $r = 1$?

$$V'(r) = \left(\tfrac{4}{3}\pi r^3\right)'$$
$$= \tfrac{4}{3}\pi (r^3)'$$
$$= \tfrac{4}{3}\pi (3r^2)$$
$$= 4\pi r^2$$

$\leftarrow$ Note: This is the formula for the surface area of a sphere! If V is in m^3, then $\frac{dV}{dr}$ has units $\frac{m^3}{m} = m^2$.

$$V'(1) = 4\pi(1)^2$$
$$= \boxed{4\pi}$$

Ex.5 The area of a circle is given by $A = \pi r^2$. Find the function for the rate of change of the area with respect to the radius r.

With respect to the radius r $\Rightarrow$ take derivative with respect to r $\Rightarrow \frac{dA}{dr}$

$$\frac{dA}{dr} = \frac{d}{dr}(\pi r^2) = \pi \frac{d}{dr}(r^2) = \boxed{2\pi r}$$

Note: What is this the formula for? The circumference!

In short: Circles (and spheres) are really cool.

Ex.3 The position of a Daedalus-class ship is given by $s(t) = -3t^2 + 2t$. What is its position when its velocity is 0?

First: when is velocity 0?

$$v(t) = -6t + 2 = 0$$
$$6t = 2$$
$$t = \frac{1}{3}$$

Next: What is position at that time?

$$s(\tfrac{1}{3}) = -3(\tfrac{1}{3})^2 + 2(\tfrac{1}{3})$$
$$= -\frac{3}{9} + \frac{2}{3}$$
$$= -\frac{1}{3} + \frac{2}{3}$$
$$= \boxed{\frac{1}{3}}$$