Lesson 9: The Product Rule

Recall:

$$\frac{d}{dx}(c) = 0$$
$$\frac{d}{dx}(x^n) = nx^{n-1}$$
$$(cf(x))' = cf'(x)$$
$$(f(x) \pm g(x))' = f'(x) \pm g'(x)$$
$$\frac{d}{dx}(\cos x) = -\sin x$$
$$\frac{d}{dx}(\sin x) = \cos x$$
$$\frac{d}{dx}(e^x) = e^x$$

* Product Rule: $\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$

Ex. 1 Differentiate $y = x^5(x + \sqrt[3]{x})$.

Method 1:

$$y = x^6 + x^{16/3}$$
$$y' = 6x^5 + \frac{16}{3}x^{13/3}$$

Method 2: $f(x) = x^5$ $\quad g(x) = x + \sqrt[3]{x}$

$f'(x) = 5x^4$ $\quad g'(x) = 1 + \frac{1}{3}x^{-2/3}$

$$y' = f'(x)g(x) + f(x)g'(x)$$
$$= (5x^4)(x + \sqrt[3]{x}) + x^5(1 + \frac{1}{3}x^{-2/3})$$
$$= 5x^5 + 5x^{13/3} + x^5 + \frac{1}{3}x^{13/3}$$
$$= 6x^5 + \frac{16}{3}x^{13/3}$$

Ex. 2 Differentiate $y = (x^2 + 2)(x^3 + e^x)$.

M1: $y = x^5 + x^2e^x + 2x^3 + 2e^x$

$y' = 5x^4 + (x^2e^x)' + 6x^2 + 2e^x$

$\hookrightarrow$ $f(x) = x^2$ $\quad g(x) = e^x$

$f'(x) = 2x$ $\quad g'(x) = e^x$

$f'(x)g(x) + f(x)g'(x)$

$= 2xe^x + x^2e^x$

$$y' = 5x^4 + 2xe^x + x^2e^x + 6x^2 + 2e^x$$

M2: $f(x) = x^2 + 2$ $\quad g(x) = x^3 + e^x$

$f'(x) = 2x$ $\quad g'(x) = 3x^2 + e^x$

$$y' = f'(x)g(x) + f(x)g'(x)$$
$$= 2x(x^3 + e^x) + (x^2+2)(3x^2 + e^x) = 2x^4 + 2xe^x + 3x^4 + x^2e^x + 6x^2 + 2e^x$$
$$= 5x^4 + 2xe^x + x^2e^x + 6x^2 + 2e^x$$

Ex. 3 Differentiate $y = \cos x(e^x + \sin x + x^3)$. Evaluate at $x=0$.

$f(x) = \cos x$ $\quad g(x) = e^x + \sin x + x^3$

$f'(x) = -\sin x$ $\quad g'(x) = e^x + \cos x + 3x^2$

$y' = f'(x)g(x) + f(x)g'(x)$

$= -\sin x\,(e^x + \sin x + x^3) + \cos x\,(e^x + \cos x + 3x^2)$

$y'|_{x=0} = -\sin(0)(e^0 + \sin(0) + 0^3) + \cos(0)(e^0 + \cos(0) + 3\cdot 0^2)$

$= 0 + 1(1+1+0)$

$= \boxed{2}$

Ex. 4 Find the x values at which $y = -x^4e^x$ has a horizontal tangent line.

Note: A horizontal tangent line has slope $m=0$.

$f(x) = -x^4$ $\quad g(x) = e^x$

$f'(x) = -4x^3$ $\quad g'(x) = e^x$

$y' = f'(x)g(x) + f(x)g'(x)$

$= -4x^3e^x + (-x^4)e^x$

$0 = -x^3e^x(4+x)$

$\Rightarrow \boxed{x = 0,\ x = -4}$

Ex. 5 Find the equation of the tan line to the curve of $y = x^2\cos x$ at $x = \pi$.

- First, find slope of tan line, so take derivative.

  $f(x) = x^2$ $\quad g(x) = \cos x$

  $f'(x) = 2x$ $\quad g'(x) = -\sin x$

  $y' = f'(x)g(x) + f(x)g'(x) = 2x\cos x - x^2\sin x$
- Evaluate at $x = \pi$: $y'|_{x=\pi} = 2\pi\cos\pi - \pi^2\sin\pi = -2\pi$
- Now, use point $(\pi, y|_{x=\pi})$ to find y-intercept.

  $y|_{x=\pi} = \pi^2\cos\pi = -\pi^2$, so $(\pi, -\pi^2)$ is on tan line.

  $y = mx + b$

  $-\pi^2 = -2\pi(\pi) + b$

  $\pi^2 = b$ $\quad \Rightarrow \boxed{y = -2\pi x + \pi^2}$

Ex.6 Differentiate $y = x^2 e^x \cos(x)$.

Problem: Multiplying three functions!

$f(x) = x^2$ $\quad g(x) = e^x \cos x$ ← Need product rule!

$f'(x) = 2x$ $\quad g'(x) = F'(x)G(x) + F(x)G'(x)$ $\quad F(x) = e^x$ $\quad G(x) = \cos x$

$= e^x \cos x + e^x(-\sin x)$ $\quad F'(x) = e^x$ $\quad G'(x) = -\sin x$

$= e^x(\cos x - \sin x)$

$y' = f'(x)g(x) + f(x)g'(x)$

$= 2x(e^x \cos x) + x^2 e^x(\cos x - \sin x)$

Ex.7 Given $f(x) = \sqrt{x}\, e^x$, find $f'(4)$.

$g(x) = \sqrt{x}$ $\quad h(x) = e^x$

$g'(x) = \frac{1}{2\sqrt{x}}$ $\quad h'(x) = e^x$

$f'(x) = g'(x)h(x) + g(x)h'(x)$

$= \frac{1}{2\sqrt{x}} e^x + \sqrt{x}\, e^x$

$f'(4) = \frac{1}{2\sqrt{4}} e^4 + \sqrt{4}\, e^4$

$= \frac{1}{4} e^4 + 2e^4$

$= \boxed{\frac{9}{4} e^4}$

Ex.8 If $s(t) = t^2 - \sqrt{t^3} \cos t$ is the position function for a particle, find the velocity function.

$v(t) = s'(t) = 2t - (t^{3/2} \cos t)'$ → $f(t) = t^{3/2}$ $\quad g(t) = \cos t$

$f'(t) = \frac{3}{2} t^{1/2}$ $\quad g'(t) = -\sin t$

$= 2t - (\frac{3}{2}\sqrt{t} \cos t - t^{3/2} \sin t)$ $\quad f'(t)g(t) + f(t)g'(t)$

$= \boxed{2t - \frac{3}{2}\sqrt{t} \cos t + t^{3/2} \sin t}$ $\quad = \frac{3}{2}\sqrt{t} \cos t - t^{3/2} \sin t$

Ex.9 Differentiate $y = (x^3 + 3x^2 + 3x + 1)(\sqrt{x} + e^x)$.

$f(x) = x^3 + 3x^2 + 3x + 1$ $\quad g(x) = \sqrt{x} + e^x$

$f'(x) = 3x^2 + 6x + 3$ $\quad g'(x) = \frac{1}{2\sqrt{x}} + e^x$

$y' = f'(x)g(x) + f(x)g'(x)$

$= (3x^2 + 6x + 3)(\sqrt{x} + e^x) + (x^3 + 3x^2 + 3x + 1)(\frac{1}{2\sqrt{x}} + e^x)$