Integration

u-sub (undoes chain rule)

Ex. $\int x e^{x^2} dx \rightarrow \left[\begin{array}{l} u = x^2 \\ du = 2x\,dx \end{array}\right] \rightarrow \frac{1}{2}\int e^u du = \frac{1}{2}e^u + C = \boxed{\frac{1}{2}e^{x^2} + C}$

Int by Parts

$$\boxed{\int u\,dv = uv - \int v\,du}$$

(undoes product rule)

- Choose u by

  Log ($u = \ln(x)$)
  Algebra ($u = x^2$)
  Trig ($u = \sin(x)$)
  Exponential ($u = e^x$)

Ex $\int x e^x dx$

$\left[\begin{array}{ll} u = x & dv = e^x dx \\ du = dx & v = e^x \end{array}\right]$

$= xe^x - \int e^x dx = \boxed{xe^x - e^x + C}$

Natural Log Integration

$$\boxed{\int \frac{1}{x} dx = \ln|x| + C}$$

Ex $\int \frac{1}{x \ln(x)} dx$

$\left[\begin{array}{l} u = \ln(x) \\ du = \frac{1}{x}\cdot dx \end{array}\right]$

$= \int \frac{1}{u} du = \ln|u| + C = \boxed{\ln|\ln(x)| + C}$

Trig Function Integrals:

$$\boxed{\begin{array}{l} \int \cos x\,dx = \sin(x) + C \\ \int \sin x\,dx = -\cos(x) + C \\ \int \sec^2(x)\,dx = \tan(x) + C \\ \int \sec(x)\tan(x)\,dx = \sec(x) + C \\ \int \csc^2(x)\,dx = -\cot(x) + C \\ \int \csc(x)\cot(x)\,dx = -\csc(x) + C \end{array}}$$

$\int \tan(x)dx = \int \frac{\sin(x)}{\cos(x)} dx$

$\left[\begin{array}{l} u = \cos(x) \\ du = -\sin(x)dx \end{array}\right]$

$= \int \frac{-du}{u} = -\ln u + C$

$\boxed{= -\ln|\cos(x)| + C}$

$(= \ln|\sec(x)| + C)$

General Solution No given data, so answer includes C.

Particular Solution Use given data to solve for C.

Ave value of f(x) over [a, b]: $\boxed{f_{ave} = \frac{1}{b-a}\int f(x)\,dx}$ ←Not derivative

Differential Equations

Growth and Decay

From calc 1: $\boxed{\frac{dy}{dt} = ky}$

(Special case of Sep of Var)

$$\int \frac{dy}{y} = \int k\,dt$$

$$\ln(y) = kt + C$$

$$y = e^{kt+C} = e^{kt} \cdot \underbrace{e^{C}}_{\text{Also constant}}$$

$$\boxed{y = Ce^{kt}}$$

Separation of Variables

$$\frac{dy}{dt} = f(t)\,g(y)$$

$$\boxed{\int (\text{all } y\text{'s})\,dy = \int (\text{all } t\text{'s})\,dt} + C$$

$$\int \frac{dy}{g(y)} = \int f(t)\,dt + C$$

First-Order Linear

$$\boxed{\underset{\substack{\uparrow \\ \text{No coeff}}}{\frac{dy}{dt}} \underset{\substack{\uparrow \\ \text{Need } +}}{+} P(t) \cdot \underset{\substack{\uparrow \\ \text{Only } y \\ \text{(linear)}}}{y} = Q(t)}$$

Integrating Factor: $\boxed{u(t) = e^{\int P(t)\,dt}}$

$$\boxed{u(t)\,y(t) = \int u(t)\,Q(t)\,dt} + C$$

Area Between Curves and Volumes

Graphing

Is it a bound or the axis of rotation?

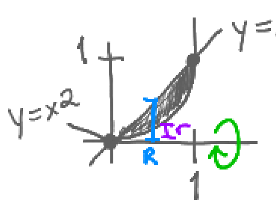
1. Graph all vertical and horizontal lines.
2. Plug these x- and y-values into the given functions to find any intersections with the boundary lines.
3. Set any functions equal to each other to find x- or y-values of intersection.
   - (i) Plot intersection points and pick an x- or y-value between the intersection points. Plug these into both functions to determine top/bottom or left/right.

Area

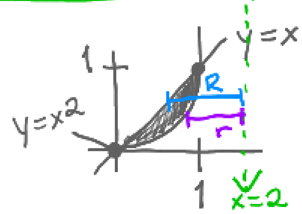
$$\boxed{\begin{array}{c}\int_a^b (\text{Top} - \text{Bottom})\,dx \\ \text{or} \\ \int_c^d (\text{Right} - \text{Left})\,dy\end{array}}$$

Solids of Revolution:

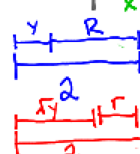
Around // to x-axis: dx
Around // to y-axis: dy



$$\int_0^1 \pi(R^2 - r^2)\,dx = \int_0^1 \pi\left((x)^2 - (x^2)^2\right)dx$$

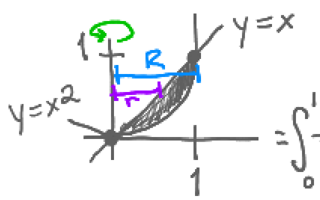


$$\int_0^1 \pi(R^2 - r^2)\,dy = \int_0^1 \pi\left((2-y)^2 - (2-\sqrt{y})^2\right)dy$$

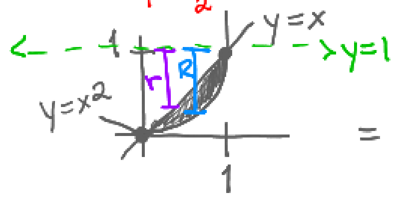


$R + y = 2 \Rightarrow R = 2 - y$

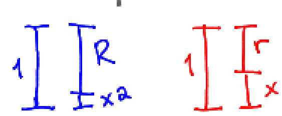
$r + \sqrt{y} = 2 \Rightarrow r = 2 - \sqrt{y}$



$$\int_0^1 \pi(R^2 - r^2)\,dy = \int_0^1 \pi\left((\sqrt{y})^2 - y^2\right)dy$$



$$\int_0^1 \pi(R^2 - r^2)\,dx = \int_0^1 \pi\left((1-x^2)^2 - (1-x)^2\right)dx$$



$1 = R + x^2$
$R = 1 - x^2$

$r + x = 1$
$r = 1 - x$

Improper Integrals

$$\int_0^1 \frac{1}{x}\,dx = \ln(x)\Big|_0^1 = \ln(1) - \underbrace{\lim_{x\to 0} \ln(x)}_{=-\infty} = \infty$$

Diverges

$\frac{1}{x}$ undefined at $x=0$

can't plug ∞ into a function; must be a limit

$$\int_2^\infty \frac{1}{x(\ln(x))^2}\,dx = -\frac{1}{\ln(x)}\Big|_2^\infty = \underbrace{\lim_{x\to\infty} -\frac{1}{\ln(x)}}_{=0} - \left(-\frac{1}{\ln(2)}\right)$$

$$= \boxed{\frac{1}{\ln(2)}}$$

(converges)

Geometric Series

$$\sum_{n=a}^{\infty} A r^n = \frac{A r^a}{1-r}$$

if $|r|<1$. (If $|r|\geq 1$, the series diverges.)

Recall rules of exponents:

$$x^{a+b} = x^a x^b$$

$$x^{a-b} = \frac{x^a}{x^b}$$

$$x^{a/b} = (x^{1/b})^a$$

$$x^{ab} = (x^a)^b$$

$$\frac{1}{x^a} = \frac{1^a}{x^a} = \left(\frac{1}{x}\right)^a$$

$$\frac{x^a}{y^a} = \left(\frac{x}{y}\right)^a$$

Functions of Several Variables

Domain

Check
1. Denominator $\neq 0$
2. Under even root ≥ 0
3. Inside log > 0

Level Curves

Set $z = k$, a constant, then solve

$$\boxed{k = f(x,y)}$$

until it is in a familiar form.

Partial Derivatives

If $z = f(x,y)$

$\boxed{\frac{\partial z}{\partial x} = f_x(x,y)}$ – treat y as constant

$\boxed{\frac{\partial z}{\partial y} = f_y(x,y)}$ – treat x as constant

Differentials

$z = f(x,y)$ and $f(a,b)$ changes to $f(a+\Delta x, b+\Delta y)$

Exact: $\Delta z = f(a+\Delta x, b+\Delta y) - f(a,b)$

Estimate: $\boxed{\Delta z \approx \underbrace{f_x(a,b)}\underbrace{\Delta x} + \underbrace{f_y(a,b)}\underbrace{\Delta y}}$

All four are numbers.

Max Error $= |f_x(a,b)|\,|\Delta x| + |f_y(a,b)|\,|\Delta y|$

Chain Rule

$z = f(x,y)$ a function of x and y

$\left.\begin{array}{l} x = x(t) \\ y = y(t) \end{array}\right\}$ functions of t

$$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}}$$

Functions of x and y

Functions of t

To evaluate at $t = \#$, plug into $x(t)$ and $y(t)$ to get x and y values. Then plug all into $\frac{dz}{dt}$.

Extrema of $f(x,y)$

- Solve $\begin{cases} f_x = 0 \\ f_y = 0 \end{cases}$ to find critical points (a,b).
- Apply the 2^{nd} DT to each critical point to classify.

Lagrange Multipliers

Optimize $f(x,y)$ subject to constraint $g(x,y) = C$.

Solve $\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ g(x,y) = C \end{cases}$ to get possible points for min/max.
(a,b)

Plug each (a,b) into $f(x,y)$ and identify min or max.

Word Problems for Min/Max Problems:

1. What function/quantity am I optimizing?
2. Do I have any constraints?

No

Find extrema with 2^{nd} DT.

Yes

How many variables are in 1?

Three

Solve the constraint for 1 variable and plug into 1. Then

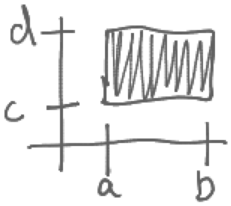
Two

Use Lagrange

Double Integrals

Over a Rectangle

$$\boxed{\int_c^d \int_a^b f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx}$$



= Volume of solid with height $f(x,y)$ over rectangle.

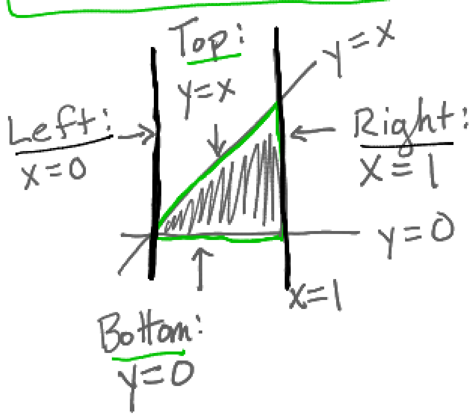
Average value of $f(x,y)$ over $[a,b] \times [c,d] = R$

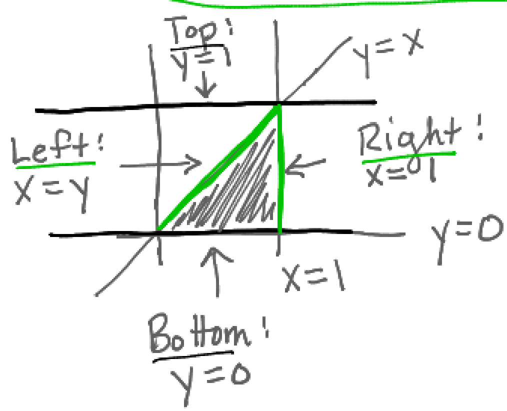
$$\boxed{f_{ave} = \frac{1}{(b-a)(d-c)} \iint_R f(x,y)\,dA}$$

Switching Limits

Graph limits of integration!

$$\boxed{\int_{a\ (\text{Left})}^{b\ (\text{Right})} \left[\int_{g_1(x)\ (\text{Bottom})}^{g_2(x)\ (\text{Top})} f(x,y)\,dy \right] dx = \int_{c\ (\text{Bottom})}^{d\ (\text{Top})} \left[\int_{h_1(y)\ (\text{Left})}^{h_2(y)\ (\text{Right})} f(x,y)\,dx \right] dy}$$





Ask yourself:

1. Can I integrate as written?
2. What are the bounds / what should I shade?

Matrices

System of Equations

$$\left[\begin{array}{c|c} \text{coefficient Matrix} & \text{constants} \end{array}\right] \leftarrow \text{Augmented Matrix}$$

Use Gauss-Jordan with row operations to get row echelon

$$\left[\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \end{array}\right]$$

or reduced row echelon

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{array}\right] \begin{array}{l} \rightarrow = x \\ \rightarrow = y \\ \rightarrow = z \end{array}$$

Consistent Independent: one solution

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{array}\right] \quad \text{or} \quad \begin{cases} x + y = 4 \\ x = 1 \end{cases}$$

z could be 0 (z = 0)

Consistent Dependent: infinite solutions (on a line)

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \text{free variable} \quad \text{or} \quad \begin{cases} x + y = 1 \\ 2x + 2y = 2 \end{cases}$$

Inconsistent: no solution

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 0 & \text{not } 0 \end{array}\right] \rightarrow 0 = \text{not zero} \quad \text{or} \quad \begin{cases} x + y = 1 \\ x + y = 2 \end{cases}$$

Matrix Operations

$$\underset{\substack{\uparrow \\ \text{constant \#}}}{k} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$$

$$A^2 = A \cdot A$$

$$AB \neq BA \text{ (only sometimes)}$$

Matrix Multiplication

$$\underbrace{\begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}}_{\substack{\text{row} \\ \text{(vector)}}} \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}}_{\substack{\text{column} \\ \text{(vector)}}} = \underbrace{a_1 b_1 + a_2 b_2 + a_3 b_3}_{\text{number}}$$

$$\underset{m \times k}{\vec{A}} \; \underset{k \times n}{B} = \underset{m \times n}{C}$$

match

dim of product

element C_{ij} of C is

$$\begin{pmatrix} \text{row } i \\ \text{of } A \end{pmatrix} \bullet \begin{pmatrix} \text{col } j \\ \text{of } B \end{pmatrix} = \#$$

Determinants

$$\boxed{\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc}$$

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

① Pick a row or column (with the most zeroes)

② Find the minor M_{ij} for all three elements in the selected row or column. by "deleting" row i and column j and find the determinant of the resulting 2×2 matrix.

③ Find the cofactors: $C_{ij} = (-1)^{i+j} M_{ij}$

The circled elements will be negative.

④ Sum $a_{ij} C_{ij}$ for the row or column.

Ex.

$$= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{31} \end{vmatrix}$$

$$= a_{11}(a_{22} a_{33} - a_{23} a_{32}) - a_{12}(a_{21} a_{33} - a_{23} a_{31}) + a_{13}(a_{21} a_{31} - a_{22} a_{31})$$

Inverses

2×2: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow \boxed{A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}$

3×3: $\left[\begin{array}{ccc|ccc} a & b & c & 1 & 0 & 0 \\ d & e & f & 0 & 1 & 0 \\ g & h & i & 0 & 0 & 1 \end{array}\right] \xrightarrow{\text{row operations}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & * & * & * \\ 0 & 1 & 0 & * & * & * \\ 0 & 0 & 1 & * & * & * \end{array}\right]$

$\underbrace{\begin{matrix} * & * & * \\ * & * & * \\ * & * & * \end{matrix}}_{= A^{-1}}$

* A is singular if $\boxed{\det(A) = 0}$ *

Use: If you write a system as

$\boxed{A\vec{x} = \vec{b}}$ $\left(\text{Ex. } \begin{cases} x + y = 1 \\ x - y = 2 \end{cases} \rightarrow \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{\vec{b}} \right)$

Then $\underbrace{A^{-1}A}_{=I}\vec{x} = A^{-1}\vec{b}$

$I\vec{x} = A^{-1}\vec{b}$

$\boxed{\vec{x} = A^{-1}\vec{b}}$ solves the system.

Eigenvalues and Eigenvectors

$$\boxed{A\vec{x} = \lambda \vec{x}}$$

① Solve $\boxed{\det(\lambda I - A) = 0}$ to find eigenvalues λ.

② Plug A and each λ $(\lambda_1, \lambda_2, \lambda_3)$ into

$$\boxed{A\vec{v_i} = \lambda_i \vec{v_i}, \quad \vec{v_i} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}}$$

and perform the multiplications to find x, y, z.

$\vec{v}_i$ is the eigenvector corresponding to eigenvalue λ_i of A.

* You will have 1 free variable$^{(t)}$ and should be able to write the remaining variable(s) in terms of the free variable$^{(t)}$ or they are 0.

Ex. $\lambda=2$ is an eigenvalue of $A=\begin{bmatrix}3&1&4\\0&2&6\\0&0&5\end{bmatrix}$

$$\begin{bmatrix}3&1&4\\0&2&6\\0&0&5\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=2\begin{bmatrix}x\\y\\z\end{bmatrix}$$

$$\begin{bmatrix}3x+y+4z\\2y+6z\\5z\end{bmatrix}=\begin{bmatrix}2x\\2y\\2z\end{bmatrix}$$

1 $\rightarrow$ $5z=2z$ $\Rightarrow z=0$

2 $\rightarrow$ $2y+\cancel{6z}^{\,z=0}=2y$

$2y=2y$

$y=y$

Free var

$y=t$

3 $\rightarrow$ $3x+\cancel{y}^{\,y=t}+\cancel{4z}^{\,z=0}=2x$

$3x+t=2x$

$x=-t$

Eigenvector: $\begin{bmatrix}-1\\1\\0\end{bmatrix}$ $\rightarrow$ can multiply by any constant, so $\begin{bmatrix}-2\\2\\0\end{bmatrix}$, $\begin{bmatrix}2\\-2\\0\end{bmatrix}$, $\begin{bmatrix}-\pi\\\pi\\0\end{bmatrix}$, $\begin{bmatrix}-101\\101\\0\end{bmatrix}$, are all eigenvectors