Unit vector in direction of $\vec{v}$: $\dfrac{\vec{v}}{|\vec{v}|}$

Scalar projection of $\vec{b}$ onto $\vec{a}$: $\text{comp}_{\vec{a}}\vec{b} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$

Vector projection of $\vec{b}$ onto $\vec{a}$: $\text{proj}_{\vec{a}}\vec{b} = \left(\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\right)\left(\dfrac{\vec{a}}{|\vec{a}|}\right)$

Line through point P_0 ($\vec{r}_0$) with direction/slope $\vec{v}$:

$\vec{r} = \vec{r}_0 + t\vec{v}$ (vector equation)

$$\begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct \end{cases}$$ (parametric eqns)

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$$ (symmetric eqns)

Line segment from $\vec{r}_0$ to $\vec{r}_1$: $\vec{r}(t) = (1-t)\vec{r}_0 + t\vec{r}_1$, $0 \le t \le 1$

Parallel Lines: same slope

Find intersection point of
$L_1: \vec{r}_1(t) = \langle x_1(t), y_1(t), z_1(t)\rangle$
$L_2: \vec{r}_2(t) = \langle x_2(t), y_2(t), z_2(t)\rangle$:

$$\begin{aligned} x_1(t) &= x_2(s) \\ y_1(t) &= y_2(s) \\ z_1(t) &= z_2(s) \end{aligned}$$

If $t = s$, the particles collide.

If no t and s satisfy this, the lines are skew.

Equation of plane through P_0 with normal vector $\vec{n}$: $a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$

Vector Function : $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$

Input : number
Output : vector

$$\frac{d\vec{r}}{dt} = \vec{r}\,'(t) = \langle f'(t), g'(t), h'(t) \rangle$$

$$\int \vec{r}\,'(t)\,dt = \left\langle \int f'(t)\,dt, \int g'(t)\,dt, \int h'(t)\,dt \right\rangle$$

Arc length function : $s(t) = \int_a^t |\vec{r}\,'(u)|\,du$

Given $\vec{r}\,'(t)$ and $\vec{r}_0 = \langle x_0, y_0, z_0 \rangle$, find $\vec{r}(t)$:

$$\vec{r}(t) = \int \vec{r}\,'(t)\,dt = \left\langle \int x'\,dt, \int y'\,dt, \int z'\,dt \right\rangle + \vec{C}$$

Find $\vec{C}$ using $\vec{r}_0$:

$$\begin{cases} x_0 = \int x'\,dt + C_1 \\ y_0 = \int y'\,dt + C_2 \\ z_0 = \int z'\,dt + C_3 \end{cases}$$

Function of Several variables : $z = f(x,y)$

Input : vector
Output : number

Volume above $D = \{(x,y)\}$ and below $z = f(x,y)$: $\iint_D f(x,y)\,dA$

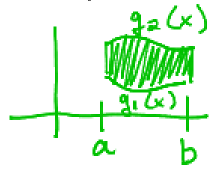
Volume with triple integral : Example : $\iiint_E dV = \iint_D \left[\int_0^{f(x,y)} dz \right] dA = \iint_D f(x,y)\,dA$

Double Integrals $\iint\limits_D f(x,y)\,dA$

Integral over a region defined by 2 vars of a function of 2 vars.

$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

Top: $g_2(x)$, Bottom: $g_1(x)$



$$\int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

Right: $h_2(y)$, Left: $h_1(y)$

Polar Coordinates $\iint\limits_D f(r,\theta)\,r\,dr\,d\theta$

$r^2 = x^2 + y^2$

$x = r\cos\theta$

$y = r\sin\theta$

$$\iint\limits_D f(x,y)\,dA = \iint\limits_D f(r\cos\theta, r\sin\theta)\,\underline{r\,dr\,d\theta}$$

Cylindrical Coordinates (r, θ, z)

$x = r\cos\theta$

$y = r\sin\theta$

$r^2 = x^2 + y^2$

$\tan\theta = \frac{y}{x}$

$z = z$

$$\iiint\limits_E f(x,y,z)\,dV = \iiint\limits_E f(r\cos\theta, r\sin\theta, z)\,\underline{r\,dz\,dr\,d\theta}$$

Spherical Coordinates (ρ, θ, ϕ)

$x = \rho\sin\phi\cos\theta$

$y = \rho\sin\phi\sin\theta$

$z = \rho\cos\phi$

$\rho^2 = x^2 + y^2 + z^2$

$$\iiint\limits_E f(x,y,z)\,dV = \iiint\limits_E f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\,\underline{\rho^2\sin\phi\,d\rho\,d\theta\,d\phi}$$

Input: vector
Output: vector

Vector Field: $\vec{F}(x,y)$ or $\vec{F}(x,y,z)$

Gradient: $\nabla f(x,y) = \langle f_x, f_y \rangle$

$$\underbrace{\nabla \underbrace{f(x,y,z)}_{\text{Function of Sev Var}}}_{\text{Vector Field}} = \langle f_x, f_y, f_z \rangle$$

Curl: $\vec{F} = \langle P, Q, R \rangle$

$$\underbrace{\text{curl}\ \underbrace{\vec{F}}_{\text{Vector Field}}}_{\text{Vector field}} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

Divergence: $\vec{F} = \langle P, Q, R \rangle$

$$\underbrace{\text{div}\ \underbrace{\vec{F}}_{\text{Vector Field}}}_{\text{Function of Sev Var}} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \nabla \cdot \vec{F}$$

Arc Length / Integral along a curve

Curve C given by $\vec{r}(t) = \langle x(t), y(t) \rangle$, $a \le t \le b$

- Length: $L = \int_C ds = \int_a^b |\vec{r}\,'(t)|\, dt$ (remember $s(t)$ is the arclength function)

- Line integral of f (function of sev. var) along curve C:

  $\int_C f(x,y)\, \underset{\substack{\uparrow \\ \text{with respect} \\ \text{to arclength } s}}{ds} = \int_a^b f(x(t), y(t))\, |\vec{r}\,'(t)|\, dt$

  $\int_C f(x,y)\, \underset{\substack{\uparrow \\ \text{w.r.t. } x}}{dx} = \int_a^b f(x(t), y(t))\, x'(t)\, dt$

  (Similar for functions of 3 variables)

  - Green's Theorem: C - closed curve, $P(x,y)$ and $Q(x,y)$ (fcn of sev var)

    $\int_C P(x,y)\, dx + Q(x,y)\, dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$

- Line integral of $\vec{F}$ (vector field) along curve C:

  $\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$

  - Stokes' Theorem: C - closed curve, positive direction (i.e. direction of $\vec{n}$), $\vec{F}$ (vector field)

    $\int_C \vec{F} \cdot d\vec{r} = \iint_S \underbrace{\text{curl } \vec{F}}_{\text{vector field}} \cdot \underbrace{d\vec{S}}_{\text{oriented surface}}$

    use flux definition

Surface Area/ Integral over a Surface

Surface given by $\vec{r}(u,v)$ with $(u,v) \in D$

- Surface Area: $\iint_S dS = \iint_D |\vec{r}_u \times \vec{r}_v| \, dA$
  - Special Case: S given by $z = f(x,y) \Rightarrow \vec{r}(x,y) = \langle x, y, f(x,y) \rangle$

    $\Rightarrow \iint_D |\vec{r}_x \times \vec{r}_y| \, dA = \iint_D \sqrt{[f_x]^2 + [f_y]^2 + 1} \, dA$

- Surface integral of f (fcn of sev var) over surface S: $\iint_S f(x,y,z) \, dS$

  $= \iint_D f(\vec{r}(u,v)) \, |\vec{r}_u \times \vec{r}_v| \, dA$

- Flux - surface integral of $\vec{F}$ (vector field) over oriented surface $\vec{S}$: $\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v) \, dA$

  need orientation to match
  i.e. if $\vec{S}$ is oriented upward, $\vec{r}_u \times \vec{r}_v$ needs $+z$ coord.

  - Divergence Theorem:
    S – boundary surface of E with positive, outward orientation
    $\vec{F}$ (vector field)

    $\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F} \, dV$

Volume / Integral over a Solid

Solid $E = \{(x,y,z) : u_1(x,y) \le z \le u_2(x,y) \;,\; (x,y) \in D\}$

or x in terms of y, z
or y in terms of x, z

- Volume $= \iiint_E dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} dz\right] dA$, etc.

- Integral of f (fcn of sev var) over a solid E : $\iiint_E f(x,y,z)\, dV$