Final Exam Review

Types of functions, equations, and operations:

Type	Input	Output
single variable function $f(x)$	scalar	scalar
function of several variables $f(x,y,z)$	point / vector	scalar
vector-valued function $\vec{r}(t)$	scalar	point / vector
vector field $\vec{F}(x,y,z)$	point/vector	point / vector
dot product	two vectors	scalar
cross product	two vectors	vector
gradient grad $f = \nabla f$	function of sev. var.	vector field
divergence div $\vec{F} = \nabla \bullet \vec{F}$	vector field	function of sev. var
curl curl $\vec{F} = \nabla \times \vec{F}$	vector field	vector field
* $\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \rangle$ or $\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$		

Types of Integrals:

- $\int_{x=a}^{x=b} f(x)\,dx$ – Calc 1 integral
- $\int \vec{v}(t)\,dt$ – The $\boxed{\text{ONLY}}$ time you should integrate a vector!
  (position, velocity, acceleration)
- $\int_C f\,ds$ – Line integral over curve C.
  – f is scalar-valued ($f(x,y)$ or $f(x,y,z)$)

  How to compute:
  Parameterize C with $\vec{r}(t)$ for $a \le t \le b$.
  Since $ds = |\vec{r}\,'(t)|\,dt$, you can compute:
  $$\int_a^b f(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt$$

- $\int_C \vec{F} \cdot d\vec{r}$ – Line integral over curve C.
  – $\vec{F}$ is a vector field ($\vec{F}(x,y) = \langle f, g \rangle$ or $\vec{F}(x,y,z) = \langle f, g, h \rangle$)
  – If C is closed, $\oint_C \vec{F} \cdot d\vec{r} = \oint_C \vec{F} \cdot \vec{T}\,ds$ is called the circulation.

  $= \int_C \vec{F} \cdot \vec{T}\,ds$ ($\vec{T}$ unit tangent)
  $= \int_C f\,dx + g\,dy$ if $\vec{F} = \langle f, g \rangle$
  $= \int_C f\,dx + g\,dy + h\,dz$ if $\vec{F} = \langle f, g, h \rangle$

  How to compute:
  – If $\vec{F}$ is conservative and C starts at A and ends at B, you can find a potential function φ such that $\vec{F} = \nabla \varphi$. Then the Fundamental Theorem of Line Integrals says:

  ① $\int_C \vec{F} \cdot d\vec{r} = \varphi(B) - \varphi(A)$ → path independent!
  $A \xrightarrow[C_1]{C_2} B$ same value

  – If C is closed and $\vec{F} = \langle f(x,y), g(x,y) \rangle$, use Green's Theorem:

  ② $\oint_C \vec{F} \cdot d\vec{r} = \iint_R (g_x - f_y)\,dA$
  (C encloses region R; $g_x - f_y$ is 2D-curl)

  – If C is closed and $\vec{F} = \langle f(x,y,z), g(x,y,z), h(x,y,z) \rangle$, use Stokes' Theorem:

  ③ $\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}\,\vec{F} \cdot d\vec{S}$
  (C is the boundary curve of S)

* If $\vec{F}$ is conservative and C is closed, ①, ②, or ③ will give you $\oint_C \vec{F} \bullet d\vec{r} = 0$.

- If $\vec{F}$ is not conservative and C is not closed, compute directly. Parameterize C with $\vec{r}(t)$ for $a \le t \le b$.

$$\int_C \vec{F} \bullet d\vec{r} = \int_C \vec{F} \bullet \underbrace{\vec{T}}_{\substack{\text{unit tangent} \\ \vec{T} = \frac{\vec{r}'(t)}{\cancel{|\vec{r}'(t)|}}}} \underbrace{ds}_{\cancel{|\vec{r}'(t)|}\, dt} = \int_a^b \vec{F}(\vec{r}(t)) \bullet \vec{r}'(t)\, dt$$

If $\vec{F} = \langle f, g \rangle$:

$$\int_C f\, dx + g\, dy = \int_a^b f(\vec{r}(t))\, x'(t)\, dt + g(\vec{r}(t))\, y'(t)\, dt$$

No dot product and <u>no</u> abs. values

If $\vec{F} = \langle f, g, h \rangle$:

$$\int_C f\,dx + g\,dy + h\,dz$$
$$= \int_a^b f(\vec{r}(t))\, x'(t)\, dt + g(\vec{r}(t))\, y'(t)\, dt + h(\vec{r}(t))\, z'(t)\, dt$$

- $\int_C \vec{F} \bullet \underbrace{\vec{n}}_{\text{unit normal}}\, ds$
  - Flux across a curve.
  - $\vec{F}$ is a vector field.

How to compute:

- If C is not closed and $\vec{F} = \langle f, g \rangle$,

$$\int_C \vec{F} \bullet \vec{n}\, ds = \int_C f\, dy - g\, dx$$

- If C is closed and $\vec{F} = \langle f, g \rangle$, use Green's Theorem:

$$\int_C \vec{F} \bullet \vec{n}\, ds = \iint_R (f_x + g_y)\, dA$$

($f_x + g_y$ is 2D-divergence)

- We did not discuss the cases for
  - closed C and $\vec{F} = \langle f, g, h \rangle$
  - not closed C and $\vec{F} = \langle f, g, h \rangle$

(We use surface integrals instead)

- $\iint_S f\, dS$ — Surface integral over S
  — f is scalar-valued

  How to compute:

  Parameterize S with $\vec{r}(u,v) = \langle x(u,v),\ y(u,v),\ z(u,v)\rangle$ over a region R in the uv-plane.
  Since $dS = |\vec{r}_u \times \vec{r}_v|\, dA$, you can compute:

  $$\iint_R f(\vec{r}(u,v))\, |\vec{r}_u \times \vec{r}_v|\, dA$$

  — For an explicitly defined surface, we can write $u = x$, $v = y$ and $z = g(x,y)$ which always gives $|\vec{r}_u \times \vec{r}_v| = |\vec{r}_x \times \vec{r}_y| = \sqrt{1 + (g_x)^2 + (g_y)^2}$. Then:

  $$\iint_S f\, dS = \iint_R f(\vec{r}(x,y))\sqrt{1 + (g_x)^2 + (g_y)^2}\, dA$$

- $\iint_S \vec{F} \bullet d\vec{S}$ — Flux and surface integral over oriented surface S

  $= \iint_S \vec{F} \bullet \underset{\substack{\uparrow \\ \text{unit normal}}}{\vec{n}}\, d\vec{S}$ — measures the net amount of $\vec{F}$ passing through the surface S in the direction of the normal to S

  How to compute:

  — Parameterize surface S with $\vec{r}(u,v)$ over a region R in the uv-plane.
  Find $\vec{r}_u \times \vec{r}_v$.
  $\vec{n}$ will point in the direction of $\vec{r}_u \times \vec{r}_v$ or in the direction $-(\vec{r}_u \times \vec{r}_v)$ based on the orientation of S

  $$\iint_S \vec{F} \bullet \vec{n}\, dS = \iint_S \vec{F}(\vec{r}(u,v)) \bullet \underset{\substack{\uparrow \\ \text{Based on orientation.}}}{(\pm(\vec{r}_u \times \vec{r}_v))}\, dA$$

  $$\vec{n} = \frac{\pm(\vec{r}_u \times \vec{r}_v)}{\cancel{|\vec{r}_u \times \vec{r}_v|}} \qquad dS = \cancel{|\vec{r}_u \times \vec{r}_v|}\, dA$$

  — If S is a surface bounding a solid region D (i.e. $(x,y,z) \in D$), use Divergence Theorem:

  $$\iint_S \vec{F} \bullet \vec{n}\, dS = \iiint_D \operatorname{div} \vec{F}\, dS$$

Limits: $\lim_{(x,y)\to(a,b)} f(x,y) = ?$

Step 1: Evaluate $f(a,b)$.
If $f(a,b)$ is defined: $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$
(and f is continuous at (a,b))

If $f(a,b)$ is not defined, f is not continuous at (a,b), but the limit might exist. Proceed to Step 2.

Step 2: Rationalize or simplify $f(x,y)$ to get $\tilde{f}(x,y)$.
(Ex. $f(x,y) = \frac{x-y}{x^2-y^2} = \frac{x-y}{(x-y)(x+y)} = \frac{1}{x+y}$,
so $\tilde{f}(x,y) = \frac{1}{x+y}$)

Evaluate $\tilde{f}(a,b)$.
If $\tilde{f}(a,b)$ is defined, then $\lim_{(x,y)\to(a,b)} f(x,y) = \tilde{f}(a,b)$.

If $\tilde{f}(a,b)$ is not defined, proceed to Step 3.

Step 3: In this course, the only remaining option for the limit to exist is to graph $f(x,y)$.

To show the limit does not exist, evaluate the limit along different curves:

$\lim_{(x,b)\to(a,b)} f(x,b)$

$\lim_{(a,y)\to(a,b)} f(a,y)$

$\lim_{(x,mx)\to(a,b)} f(x,mx)$

$\lim_{(x,mx^2)\to(a,b)} f(x,mx^2)$

$\lim_{(my^2,y)\to(a,b)} f(my^2,y)$

At least two of these five limits should be different which means $\lim_{(x,y)\to(a,b)} f(x,y)$ does not exist.