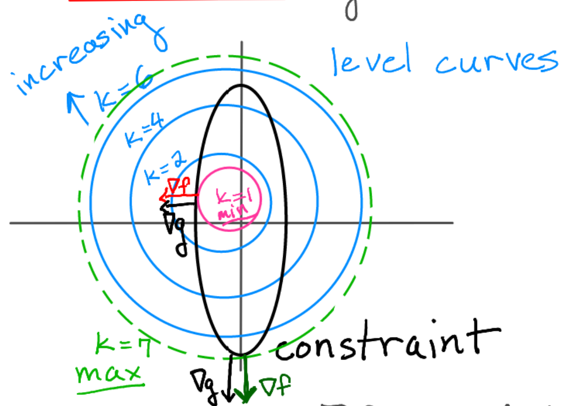
§ 15.8 Lagrange Multipliers

Now we focus on finding minimum and maximum values of an objective function f subject to a constraint g.

Looking at level curves:

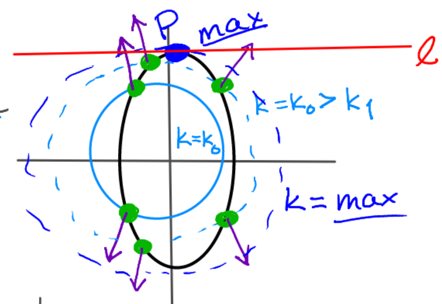


$z = f(x,y) \rightarrow$ level curves $k = f(x,y)$

$0 = g(x,y) \rightarrow$ constraint equal to a constant 0

The minimum and maximum values occur where ∇f and ∇g are parallel.

Why? First, draw the constraint.
Next, draw any level curve that intersects the constraint.
We can calculate ∇f at these points.
As we increase k, the level curves will be "moving" along ∇f.
Once we increase k to the point where the constraint and level curve are tangent, we cannot increase k and still intersect the constraint.
Line ℓ is tangent to both f and g at P, so ∇f and ∇g are orthogonal to ℓ at P which means they are parallel to each other.
We use this fact to determine the min and max.



Thm Lagrange multipliers

Let f be an objective function and $g = 0$ be a constraint. Suppose f has an extreme value, i.e. a min or max, at P.
Then if $\nabla g \neq 0$ at P, we have $\nabla f = \lambda \nabla g$ at P.

λ: Lagrange multiplier

Note: The theorem tells us <u>if</u> there is a min or max, $\nabla f = \lambda \nabla g$. However, $\nabla f = \lambda \nabla g$ does <u>not</u> mean there is a min or max at this point.
As with absolute extrema, we evaluate f at all points where $\nabla f = \lambda \nabla g$ and identify the min and max.

Method: ① Solve the system $\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y) = 0 \end{cases}$

If working with $f(x,y)$, we find points (x,y).
If working with $f(x,y,z)$, " " (x,y,z).
And so on.

② Evaluate f at all points found in ① to identify min and max f values.

Ex.1 Find the absolute min and max of $f(x,y) = x - y$ subject to the constraint $x^2 + y^2 - 3xy = 20$.

Note: We cannot easily solve $x^2 + y^2 - 3xy = 20$ for x or y. If we have a constraint like $x^2 + y = 20$, we can write $y = 20 - x^2$ and plug this into $f(x,y)$ to get $f(x) = x - 20 + x^2$ which is a <u>calc 1</u> optimization problem. Instead, we need to use Lagrange multipliers.

$\nabla f = \langle f_x, f_y \rangle = \langle 1, -1 \rangle$
$g(x,y) = x^2 + y^2 - 3xy - 20$
$\nabla g = \langle 2x - 3y, 2y - 3x \rangle$

Solve $\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y) = 0 \end{cases} \Rightarrow \begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ x^2 + y^2 - 3xy - 20 = 0 \end{cases}$

$$\Rightarrow \begin{cases} ① \; 1 = \lambda (2x - 3y) & \Rightarrow \lambda \neq 0 \text{ and } 2x - 3y \neq 0 \\ ② \; -1 = \lambda (2y - 3x) & \Rightarrow \lambda \neq 0 \text{ and } 2y - 3x \neq 0 \\ ③ \; x^2 + y^2 - 3xy - 20 = 0 \end{cases}$$

Then ① $\Rightarrow \lambda = \frac{1}{2x - 3y}$ and ② $\Rightarrow \lambda = \frac{-1}{2y - 3x}$

$$\frac{1}{2x - 3y} = \frac{-1}{2y - 3x}$$
$$2y - 3x = -2x + 3y$$

$$-y = x$$

We solved both equations for λ to relate x and y. Now, we can plug this relation into ③ to get values for x or y.

$$③ \quad x^2 + y^2 - 3xy - 20 = 0$$
$$(-y)^2 + y^2 - 3(-y)y - 20 = 0$$
$$y^2 + y^2 + 3y^2 - 20 = 0$$
$$5y^2 = 20$$
$$y = \pm 2$$

Since $x = -y$, $\quad y = 2 \Rightarrow x = -2$ $\quad$ and $\quad y = -2 \Rightarrow x = 2$

Now, we evaluate $f(x,y) = x - y$ at $(-2,2)$ and $(2,-2)$:

(x,y)	$f(x,y) = x - y$	
$(-2,2)$	$f(-2,2) = -2-2 = \boxed{-4}$	min
$(2,-2)$	$f(2,-2) = 2-(-2) = \boxed{4}$	max

Ex.2 Find the absolute min and max of $f(x,y) = xyz$ subject to $x^2 + 2y^2 + 4z^2 = 9$.

$$\nabla f = \langle f_x, f_y, f_z \rangle = \langle yz, xz, xy \rangle$$
$$g(x,y) = x^2 + 2y^2 + 4z^2 - 9$$
$$\nabla g = \langle g_x, g_y, g_z \rangle = \langle 2x, 4y, 8z \rangle$$

Solve
$$\begin{cases} yz = \lambda(2x) = 2\lambda x & ① \\ xz = \lambda(4y) = 4\lambda y & ② \\ xy = \lambda(8z) = 8\lambda z & ③ \\ x^2 + 2y^2 + 4z^2 = 9 & ④ \end{cases}$$

If $x \neq 0$: $\quad ① \Rightarrow \lambda = \frac{yz}{2x}$

If $y \neq 0$: $\quad ② \Rightarrow \lambda = \frac{xz}{4y}$

$$\frac{yz}{2x} = \frac{xz}{4y}$$
$$\Rightarrow 4y^2 z = 2x^2 z$$
$$0 = 2x^2 z - 4y^2 z$$
$$0 = \underbrace{2z}_{z=0}\underbrace{(x^2 - 2y^2)}_{x^2 = 2y^2}$$
$$\Rightarrow$$

If $z \neq 0$: $\quad ③ \Rightarrow \lambda = \frac{xy}{8z}$

$$\frac{xy}{8z} = \frac{yz}{2x}$$
$$2x^2 y = 8yz^2$$
$$0 = 2y\underbrace{(4z^2 - x^2)}_{x^2 = 4z^2}$$

Now, ④ $\Rightarrow x^2 + 2\left(\frac{x^2}{2}\right) + 4\left(\frac{x^2}{4}\right) - 9 = 0$

$$3x^2 = 9$$
$$x = \pm\sqrt{3}$$

(∗) $y^2 = \frac{x^2}{2}$, $z^2 = \frac{x^2}{4}$

$x = -\sqrt{3}$, $y = \pm\frac{\sqrt{3}}{\sqrt{2}}$, $z = \pm\frac{\sqrt{3}}{2}$

$x = \sqrt{3}$, $y = \pm\frac{\sqrt{3}}{\sqrt{2}}$, $z = \pm\frac{\sqrt{3}}{2}$

If $x=0$: ① $yz = 0$
② $0 = 4\lambda y$
③ $0 = 8\lambda z$

$\Rightarrow y = 0, \lambda = 0$ OR $z = 0, \lambda = 0$ OR $y = z = 0$ Does not satisfy constraint

④ $4z^2 = 9$, $z = \pm\frac{3}{2}$ ④ $2y^2 = 9$, $y = \pm\frac{3}{\sqrt{2}}$

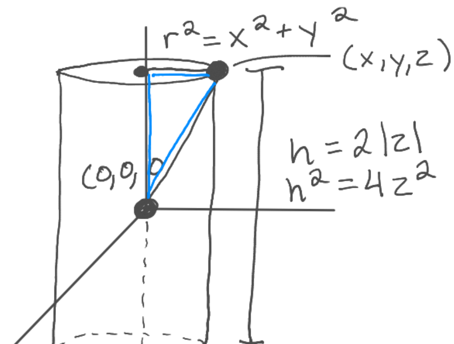
Similar if $y=0$ or $z=0$.
However! We do not need to find and check all of these points because if $x=0$, $y=0$, or $z=0$, then $f(x,y,z) = xyz = 0$.

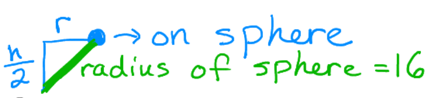
Now, we need to check the points from (∗). Since we are looking for the min and max values and not the points where they occur, we see that for all 8 points (∗), $|xyz| = \left|\sqrt{3}\cdot\frac{\sqrt{3}}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2}\right| = \frac{3\sqrt{3}}{2\sqrt{2}}$.
Then $f(x,y,z) = \pm\frac{3\sqrt{3}}{2\sqrt{2}}$ depending on the signs of x, y, z.

$$\boxed{\begin{aligned} \text{Max} &= \frac{3\sqrt{3}}{2\sqrt{2}} \\ \text{Min} &= -\frac{3\sqrt{3}}{2\sqrt{2}} \end{aligned}}$$

Ex.3 Find the dimensions of the right circular cylinder of maximum volume that can be inscribed in a sphere of radius 16.

Objective: maximize volume
Constraint: $x^2 + y^2 + z^2 = 16^2 = 256$



From the picture,


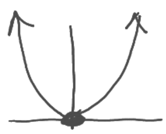
$\left(\frac{h}{2}\right)^2 + r^2 = 16^2$

so in terms of r and h, we have $V(r,h) = \pi r^2 h$
$$g(r,h) = \frac{h^2}{4} + r^2 - 256$$

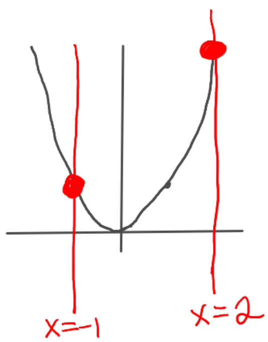
Answer: $r = \frac{16\sqrt{6}}{3}$, $h = \frac{32\sqrt{3}}{3}$

Note: When we are finding <u>absolute</u> extrema, we do <u>NOT</u> use the 2^{nd} Derivative Test to classify mins and maxes because points (a,b) that satisfy $\begin{cases} \nabla f(a,b) = \lambda \nabla g(a,b) \\ g(a,b) = 0 \end{cases}$ are <u>not necessarily critical points</u> of f (i.e. $\nabla f(a,b) \neq \vec{0}$).

Recall calc 1 (absolute extrema on an interval)



$f(x) = x^2$
$f'(x) = 2x = 0$
$x = 0$ is the critical number
$f''(x) = 2 \Rightarrow \underbrace{2^{nd} DT}_{f''(0) > 0}$ gives local/relative min @ $x=0$



Now, restrict $f(x)$ to $-1 \leq x \leq 2$

We still have the critical number $x=0$ in the interval, but we also have to check the boundaries $x=-1$ and $x=2$:

x	$f(x) = x^2$	
0	0	$\rightarrow$ Absolute Min @ $x=0$
-1	1	
2	4	$\rightarrow$ Absolute Max @ $x=2$

Since $f''(x) = 2 > 0$ for <u>all</u> x, the second derivative test only works for <u>local</u>/relative extrema. The same is true for greater dimensions.