§13.6 Cylinders and Quadratic Surfaces

Def. A trace is the set of points at which a surface intersects a plane that is parallel to one of the coordinate planes.

xy-trace: intersection with $z=0$ (the xy-plane)
yz-trace: intersection with $x=0$ (the yz-plane)
xz-trace: intersection with $y=0$ (the xz-plane)

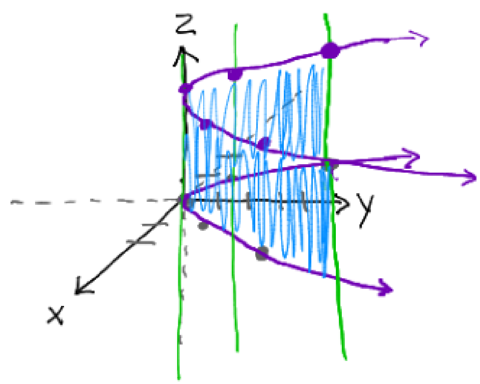
Note: Intersection with coordinate axes

x-axis: Set $y=0$ and $z=0$
y-axis: Set $x=0$ and $z=0$
z-axis: Set $x=0$ and $y=0$

Def. A cylinder is a surface that consists of all lines that are parallel to a given line and pass through a given curve.

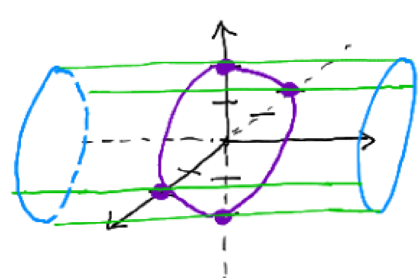
Ex. $y = x^2$

Because there is no restriction on z, this surface consists of all lines parallel to the z-axis that pass through the curve $y = x^2$ (in the xy-plane).



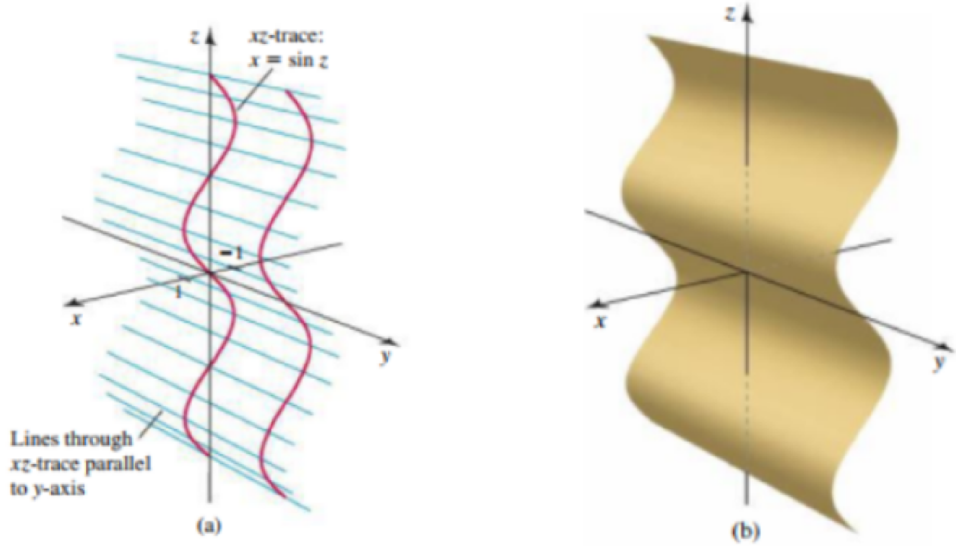
Ex. $x^2 + z^2 = 4$

No restriction on y, so parallel to y-axis through the circle $x^2+z^2=4$ (in the xz-plane).



Ex. $x - \sin z = 0$

See Fig. 13.82 (below) from Pg. 857 of the text.



Def. Quadratic surfaces have the general form

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0$$

where not all of A, B, C, D, E, and F are 0.

We will focus on a smaller class where $D = E = F = 0$.

Note: $ax + by + cz = d$ is a plane

all variables linear $\rightarrow$ Not a quadratic surface.

$x^2 + y^2 + z^2 = r^2$ is a sphere

all squared $\longrightarrow$ Quadratic Surface

Also, some cylinders are quadratic surfaces (like $y = x^2$ and $x^2 + z^2 = 4$, but $x = \sin z$ is not).

Def. Ellipsoids have the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

All traces are ellipses. A sphere when $a=b=c$.

Ellipsoid

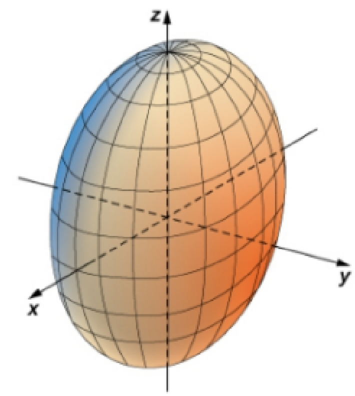
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

Traces

In plane $z=p$: an ellipse
In plane $y=q$: an ellipse
In plane $x=r$: an ellipse

Ellipsoid

If $a=b=c$, then this surface is a sphere.



Elliptic paraboloids have the form $\frac{z}{c}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$.

Traces in two directions are parabolas. In this case, when $y=0$: $\frac{z}{c}=\frac{x^2}{a^2}$ and when $x=0$: $\frac{z}{c}=\frac{y^2}{b^2}$.

Traces in one direction are ellipses. In this case, when $z=c$: $1=\frac{x^2}{a^2}+\frac{y^2}{b^2}$.

When $z=0$, the "ellipse" is the point $(0,0,0)$.

Enter as $\frac{x^2}{a^2}+\frac{y^2}{b^2}=0$

Elliptic Paraboloid

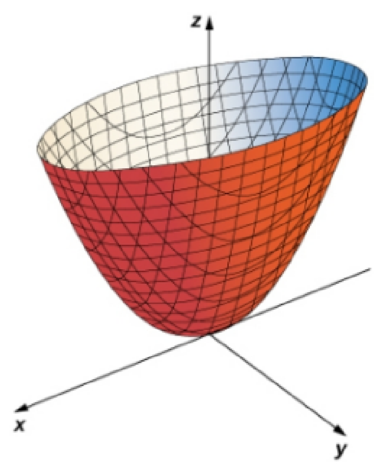
$$z=\frac{x^2}{a^2}+\frac{y^2}{b^2}$$

Traces

In plane $z=p$: an ellipse — Elliptic
In plane $y=q$: a parabola
In plane $x=r$: a parabola

Paraboloid

The axis of the surface corresponds to the linear variable.



Hyperbolic paraboloids have the form $\frac{z}{c}=\frac{x^2}{a^2}-\frac{y^2}{b}$

Traces in <u>two</u> directions are parabolas. In this case, when $y=0: \frac{z}{c}=\frac{x^2}{a^2}$ and when $x=0: \frac{z}{c}=-\frac{y^2}{b^2}$.

Traces in <u>one</u> direction are hyperbolas. In this case, when $z=c$: $1=\frac{x^2}{a^2}-\frac{y^2}{b^2}$.

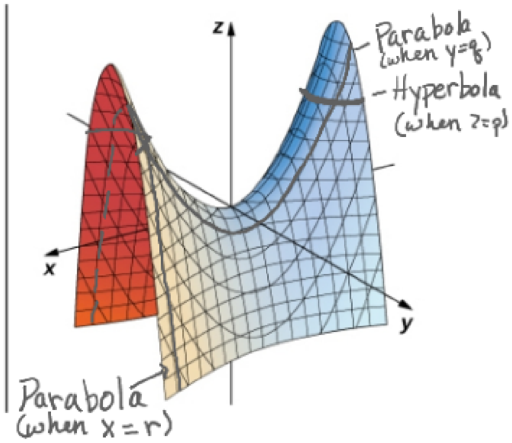
Hyperbolic Paraboloid

$$z=\frac{x^2}{a^2}-\frac{y^2}{b^2}$$

Traces

In plane $z=p$: a hyperbola — Hyperbolic
In plane $y=q$: a parabola
In plane $x=r$: a parabola — Paraboloid

The axis of the surface corresponds to the linear variable.



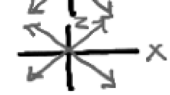
(Elliptic) cones have the form $\frac{z^2}{c^2}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$.

Traces in <u>two</u> directions are two lines. In this case,

Enter the squared versions, not absolute values

when $x=0$: $\frac{z^2}{c^2}=\frac{y^2}{b^2} \Rightarrow \frac{|z|}{|c|}=\frac{|y|}{|b|} \Rightarrow$

$y=0$: $\frac{z^2}{c^2}=\frac{x^2}{a^2} \Rightarrow \frac{|z|}{|c|}=\frac{|x|}{|a|} \Rightarrow$



Traces in <u>one</u> direction are ellipses. In this case, when $z=c$: $1=\frac{x^2}{a^2}+\frac{y^2}{b^2}$. Again, when $z=0$, the "ellipse" is the point $(0,0,0)$. This point is the center of the cone.

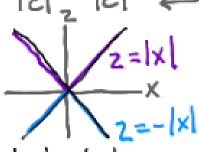
Enter as $\frac{x^2}{a^2}+\frac{y^2}{b^2}=0$.

Elliptic Cone

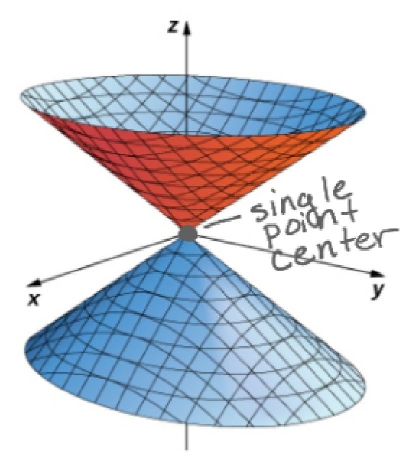
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$$

Traces

In plane $z=p$: an ellipse — Elliptic
In plane $y=q$: a hyperbola
In plane $x=r$: a hyperbola
In the xz – plane: a pair of lines that intersect at the origin ← $\frac{|z|}{|c|}=\frac{|x|}{|c|}$
In the yz – plane: a pair of lines that intersect at the origin



The axis of the surface corresponds to the variable with a negative coefficient. The traces in the coordinate planes parallel to the axis are intersecting lines.



Hyperboloids of one sheet have the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

Traces in two directions are hyperbolas. In this case, when $y=0$: $\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1$

when $x=0$: $\frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$

Traces in one direction are ellipses. In this case, when $z=0$: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. The center of this ellipse is the center of the surface.

Hyperboloid of One Sheet

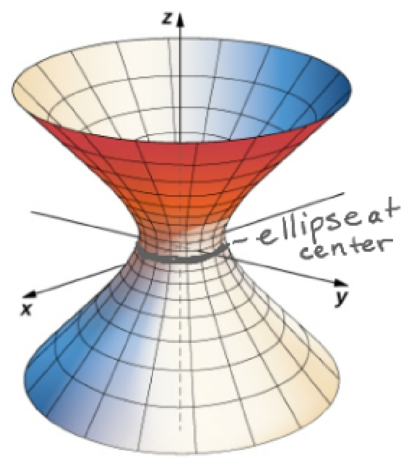
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

Traces

In plane $z = p$: an ellipse

$\left.\begin{array}{l}\text{In plane } y = q\text{: a hyperbola} \\ \text{In plane } x = r\text{: a hyperbola}\end{array}\right\}$ Hyperboloid

In the equation for this surface, two of the variables have positive coefficients and one has a negative coefficient. The axis of the surface corresponds to the variable with the negative coefficient.



Note: Center could be shifted.

For instance, $(x-2)^2 + (y+1)^2 - (z+2)^2 = 1$.

To determine if cone, hyperboloid of one sheet or hyperboloid of two sheets, check

$\left.\begin{array}{ll}x = 2: & \text{hyperbola} \\ y = -1: & \text{hyperbola} \\ z = -2: & \text{ellipse (circle)}\end{array}\right\}$ hyperboloid of one sheet.

Hyperboloids of two sheets have the form

$$-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

Traces in two directions are hyperbolas.

In this case, $x=0$: $-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

$y=0$: $-\frac{x^2}{a^2}+\frac{z^2}{c^2}=1$.

Traces in one direction are ellipses.

$$\frac{z^2}{c^2}-1=\frac{x^2}{a^2}+\frac{y^2}{b^2}$$

$\Rightarrow$ If $\frac{z^2}{c^2}-1<0$, $-c<z<c$, the the equation is not valid, so there are no traces.

$\Rightarrow$ When $z=\pm c$, $0=\frac{x^2}{a^2}+\frac{y^2}{b^2}$, so the "vertices" of the sheets are $(0,0,c)$ and $(0,0,-c)$, and the distance between the sheets is $2c$.

Hyperboloid of Two Sheets

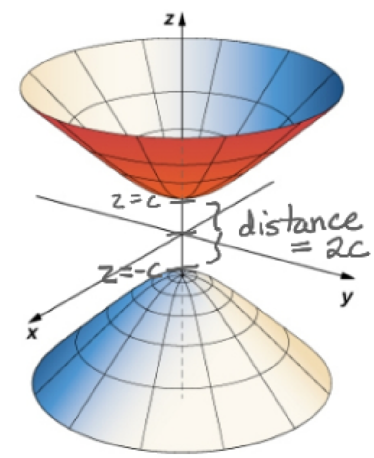
$$\frac{z^2}{c^2}-\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

Traces

In plane $z=p$: an ellipse or the empty set (no trace)

$\left.\begin{array}{l}\text{In plane } y=q\text{: a hyperbola}\\ \text{In plane } x=r\text{: a hyperbola}\end{array}\right\}$ hyperboloid

In the equation for this surface, two of the variables have negative coefficients and one has a positive coefficient. The axis of the surface corresponds to the variable with a positive coefficient. The surface does not intersect the coordinate plane perpendicular to the axis.



Ex. $(x-2)^2-(y-7)^2-(z+3)^2=1$

$x=2$: $-(y-7)^2-(z+3)^2=1$

$(y-7)^2+(z+3)^2=-1$ Not possible (No trace)

$y=7$: Hyperbola

$z=-3$: Hyperbola

Hyperboloid of two sheets

§14.1 Vector-Valued Functions

Def. Vector-valued functions have a single, scalar independent variable (often t, s, or θ) and multiple dependent variables forming a vector.

Input: Scalar
Output: vector

$$\vec{r}(t) = \langle f(t), g(t), h(t)\rangle = f(t)\,\vec{i} + g(t)\,\vec{j} + h(t)\,\vec{k}$$

f, g, h are scalar-valued functions.

We can write $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$.

In either case, f, g, and h or x, y, and z are the component functions of $\vec{r}$.

Note: The equation of a line is a vector-valued function where all component functions are linear (in t).

Def. The domain of $\vec{r}(t)$ is the largest set of values of t where all the component functions defined.

Common domain issues:
1. Division by 0
2. Negative inside even root.
3. Negative or 0 inside log.

Def. The positive orientation of a curve is the direction of the curve generated by increasing values of the parameter.
Negative orientation is with decreasing values of the parameter.

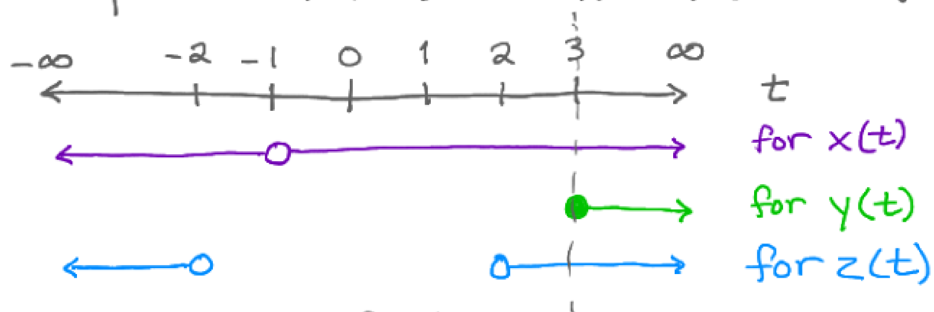
Ex. 1 Find the domain of $\vec{r}(t) = \left\langle \underset{x(t)}{\frac{1}{t+1}}, \underset{y(t)}{\sqrt{t-3}}, \underset{z(t)}{\ln(t^2-4)} \right\rangle$.

① Division by 0 possible in $x(t)$.
Need $t+1 \neq 0 \Rightarrow t \neq -1$

② Negative under square root possible in $y(t)$.
Need $t-3 \geq 0 \Rightarrow t \geq 3$

③ Nonpositive inside log possible in $z(t)$.
Need $t^2-4>0 \Rightarrow t^2>4 \Rightarrow t<-2$ or $t>2$

④ Graph t values on number line.



Need all satisfied, so the domain is
$\{t : 3 \leq t < \infty\}$

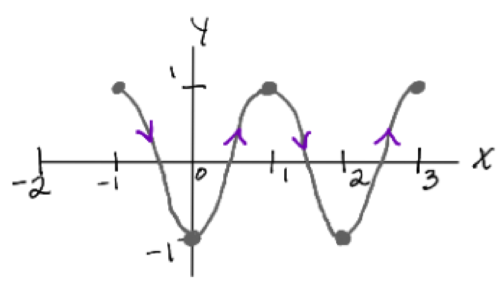
Read: "t such that t is greater than or equal to 3 and less than infinity"

Ex. 2 Graph the curve of $\vec{r}(t) = \langle t+1, \cos(\pi t)\rangle$ for $-2 \leq t \leq 2$. Indicate positive orientation.

① Make a table:

t	$x = t+1$	$y = \cos(\pi t)$
-2	-1	1
-1	0	-1
0	1	1
1	2	-1
2	3	1

② Graph (x,y) points



③ Add arrows to indicate orientation

Ex.3 Find the points at which the curve $\vec{r}(t) = t\,\vec{i} + (2t - t^2)\,\vec{k}$ intersects the surface $z = x^2 + y^2$.
(This surface is an elliptic paraboloid.)

Points where the curve and surface intersect must satisfy both functions.

① From the curve, we know
$$x = t \qquad y = 0 \qquad z = 2t - t^2$$

② Now, plug these into
$$z = x^2 + y^2$$
$$2t - t^2 = t^2$$
$$0 = 2t^2 - 2t$$
$$0 = \underbrace{2t}_{t=0}\,\underbrace{(t-1)}_{t=1}$$

③ We have t values, but we need points:

	$t=0$	$t=1$
$x = t$	$x = 0$	$x = 1$
$y = 0$	$y = 0$	$y = 0$
$z = 2t - t^2$	$z = 0$	$z = 1$

The points of intersection are $\boxed{(0,0,0) \text{ and } (1,0,1)}$

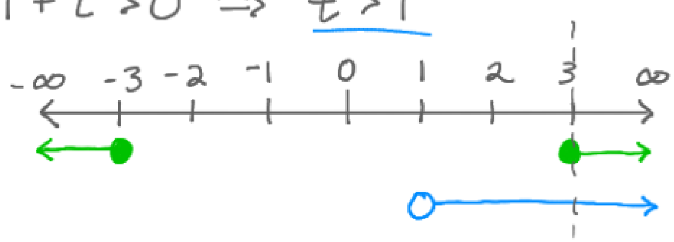
Ex.4 Find the domain of $\vec{r}(t) = \langle \underbrace{\sqrt[3]{t+1}}_{-\infty < t < \infty},\ \ln(-1+t),\ \sqrt{t^2 - 9} \rangle$.

① Division by 0 not possible in $x, y,$ or z.

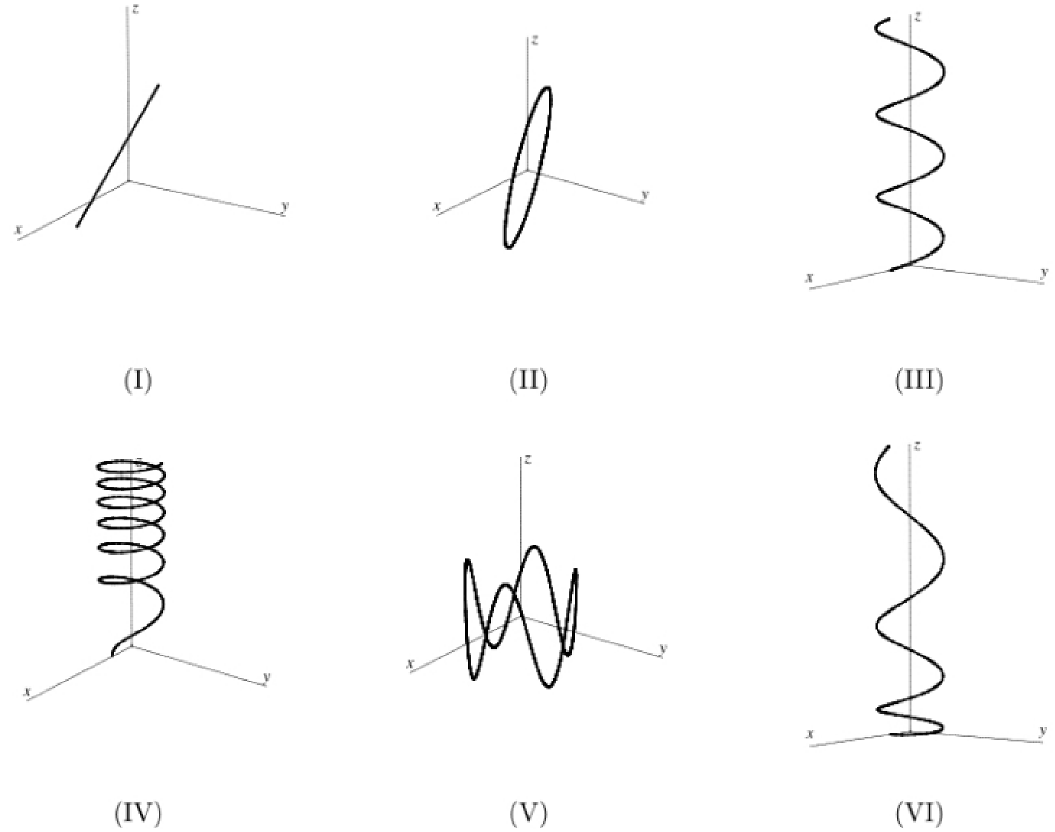
② Negative under even root possible in $z(t)$.
<u>Need</u>: $t^2 - 9 \geq 0 \Rightarrow t^2 \geq 9 \Rightarrow t \leq -3$ or $t \geq 3$

③ Nonpositive possible inside log in $y(t)$.
<u>Need</u>: $-1 + t > 0 \Rightarrow t > 1$

④ <u>Graph</u>:



$\boxed{\{t : 3 \leq t < \infty\}}$



Ex. 5 For each vector-valued function (a)–(e), determine which curve (I)–(VI) is its graph.

(a) $\vec{r}(t) = \langle \underset{x}{\cos t}, \underset{y}{\sin t}, \underset{z}{t} \rangle$

Relate the variables.

Since $z = t$, $x = \cos z$ in planes parallel to xz-plane,
and $y = \sin z$ in " " yz-plane.

For every 2π increase in t, z increases by 2π and x and y complete one cycle through $\cos z$ and $\sin z$, so we should have even, equally spaced spirals.

$\boxed{\text{III}}$

(b) $\vec{r}(s) = \langle \underset{x}{\cos s}, \underset{y}{\sin s}, \underset{z}{\sin 4s} \rangle$

$0 \le s \le \frac{\pi}{2}$ (period for $\sin 4s$) $\Rightarrow \underset{s=0}{(1,0,0)} \to \underset{s=\frac{\pi}{4}}{\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right)} \to \underset{s=\frac{\pi}{2}}{(0,1,0)}$

The period for cos s and sin s is 2π while the period for sin 4s is $\frac{\pi}{2}$.

This means to trace the whole graph 1 time, a particle travels between z values −1 and 1 4 times, so we should have 4 peaks and valleys for z.

$\boxed{\text{V}}$

(c) $\vec{r}(s) = \langle \cos s, \sin s, 4\sin s \rangle$

The period for all coordinates x, y, and z is 2π.

The bounds for the coordinates are

$-1 \le x \le 1$
$-1 \le y \le 1$ so we need a graph that traces
$-4 \le z \le 4$

through x, y, and z periodically (closed curve) and is stretched in the z-direction.

$\boxed{\text{II}}$

(d) $\vec{r}(u) = \langle \cos u^3, \sin u^3, u^3 \rangle$

As in (a), we can relate the variables since $z = u^3$. Then $x = \cos z$ and $y = \sin z$.

This gives the same curve as $\langle \cos t, \sin t, t \rangle$.

However, $t = u^3$, so the interval for u is much smaller to draw the same number of spirals.

$\boxed{\text{III}}$

(e) $\vec{r}(u) = \langle 3 + 2\cos u, 1 + 4\cos u, 2 + 5\cos u \rangle$

The easiest thing here is to do a change of variables: $t = \cos u$ with $-1 \leq t \leq 1$.

Then we see that this curve is a line.

$\boxed{\text{I}}$