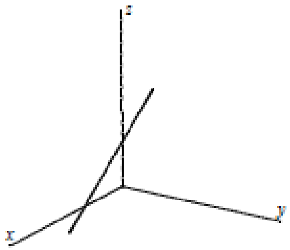
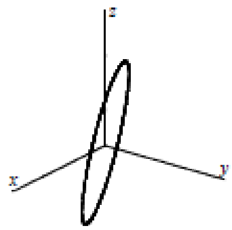
Vector-Valued Functions

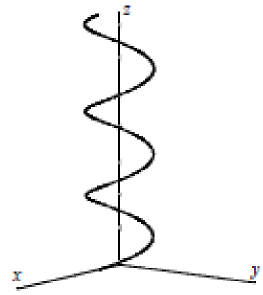
1. Here are several curves.



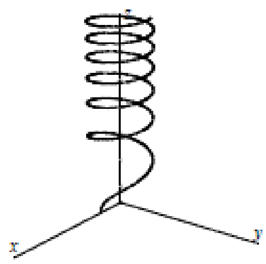
(I)



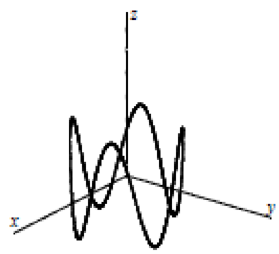
(II)



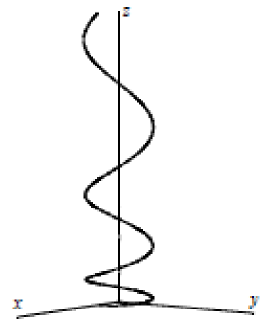
(III)



(IV)



(V)



(VI)

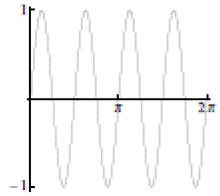
Find the curve parameterized by each vector-valued function.

(a) $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$.

Solution. Remember that we visualize this by imagining a particle in space whose position at time t is $(\cos t, \sin t, t)$. Let's first just think about what the particle's x- and y-components are doing. (One way to visualize this is to imagine looking down on the particle from above; then you can't really see what its height is, so you're looking only at what its x- and y-components are doing.) We know that $x = \cos t$, $y = \sin t$ traces out a circle (the unit circle $x^2 + y^2 = 1$), repeating its path every 2π. Since the z component is just t, each time the particle's x- and y-components trace out a circle, the particle rises by 2π. This matches picture $\boxed{\text{(III)}}$.

(b) $\vec{r}(s) = \langle \cos s, \sin s, \sin 4s \rangle$.

Solution. The x- and y-components here are the same as in (a). However, from time $s = 0$ to $s = 2\pi$, the particle's height is given by $\sin 4s$, whose graph looks like this:



That is, each time the x- and y-components of the particle make a loop, the height should rise and fall four times. This matches $\boxed{\text{(V)}}$.

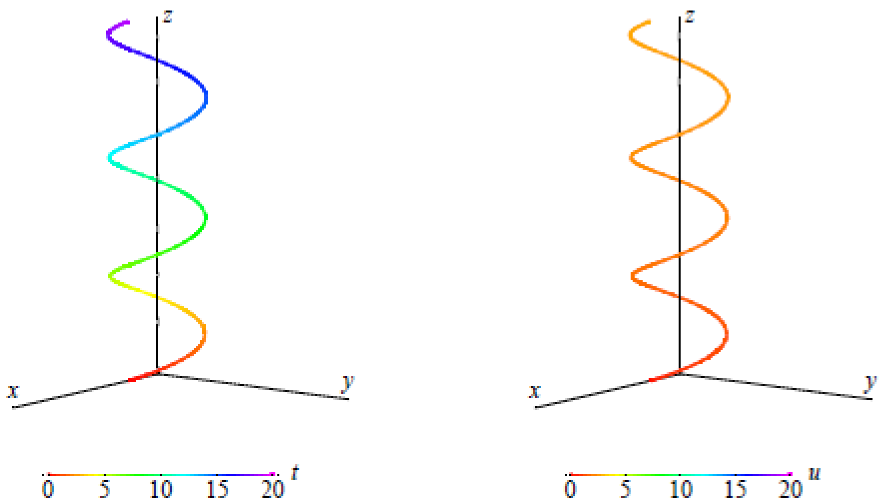
(c) $\vec{r}(s) = \langle \cos s, \sin s, 4\sin s\rangle$.

Solution. Using the same line of reasoning as in (a) and (b), we see that this is $\boxed{\text{(II)}}$. Notice that the entire curve appears to lie in a single plane. This is indeed the case, which is easy to check from the equations: if $x = \cos s$, $y = \sin s$, and $z = 4\sin s$, then z is always equal to $4y$, so the curve sits in the plane $z = 4y$.

(d) $\vec{r}(u) = \langle \cos u^3, \sin u^3, u^3\rangle$.

Solution. Observe that the particle's position at time $t = u^3$ is just $(\cos t, \sin t, t)$, which is exactly the same as in (a). Thus, the curve traced out by this function is again $\boxed{\text{(III)}}$. The difference between this function and the one in part (a) is the time it takes the particle to reach a given point on the curve. (Another way of saying the same thing is that a particle traveling according to the function in (a) and a particle traveling according to the function here travel the same path, but they go at different speeds.)

Here is a visual illustration:



The left picture shows $\vec{r}(t) = \langle \cos t, \sin t, t\rangle$, with the value of t at a particular point on the curve indicated by color. A particle traveling according to this parameterization reaches the top point at time $t = 6\pi$ ($6\pi \approx 18.8$, so this point is colored purple).

On the other hand, a particle traveling according to the parameterization $\vec{r}(u) = \langle \cos u^3, \sin u^3, u^3\rangle$ reaches the top point much more quickly (in fact, at time $u = \sqrt[3]{6\pi} \approx 2.7$), as shown in the right picture.

(e) $\vec{r}(u) = \langle 3 + 2\cos u, 1 + 4\cos u, 2 + 5\cos u \rangle$.

Solution. This is simpler than it looks. At time $t = \cos u$, the particle is at the point $(3 + 2t, 1 + 4t, 2 + 5t)$. You should recognize this as parameterizing a line, so the correct picture is $\boxed{\text{(I)}}$.

Notice that, when $u = 0$, the particle is at $(5, 5, 7)$; when $u = \pi$, the particle is at $(1, -3, -3)$. When $u = 2\pi$, the particle is back at $(5, 5, 7)$. Since $t = \cos u$ oscillates between 1 and -1 forever, the particle just travels back and forth along the line segment between $(5, 5, 7)$ and $(1, -3, -3)$.