§14.2 Calculus of Vector-Valued Functions

* To integrate and differentiate vector-valued functions, we integrate and differentiate the component functions as we did in previous calc courses.

Ex.1 Differentiate $\vec{r}(t) = \langle \underbrace{te^{-t}}_{x(t)}, \underbrace{\ln(t^2+1)}_{y(t)}, \underbrace{\frac{\sin t}{t}}_{z(t)} \rangle$.

$$\vec{r}\,'(t) = \frac{d}{dt}[\vec{r}(t)] = \langle x'(t), y'(t), z'(t) \rangle$$

$$x'(t) = \underbrace{\frac{d}{dt}[te^{-t}]}_{\text{Product Rule}} = \frac{d}{dt}[t]\cdot e^{-t} + t\cdot \underbrace{\frac{d}{dt}[e^{-t}]}_{\text{Chain Rule}}$$

$$= (1)e^{-t} + t(-e^{-t}) = e^{-t} - te^{-t}$$

$$y'(t) = \underbrace{\frac{d}{dt}[\ln(t^2+1)]}_{\text{Chain Rule}} = \frac{2t}{t^2+1}$$

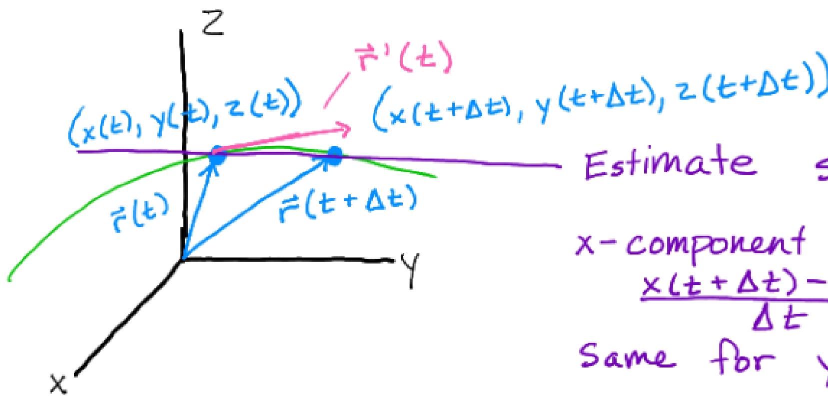
$$z'(t) = \underbrace{\frac{d}{dt}\left[\frac{\sin t}{t}\right]}_{\text{Quotient Rule}} \quad \left(= \underbrace{\frac{d}{dt}[t^{-1}\cdot \sin t]}_{\text{Product Rule}}\right)$$

$$= \frac{\frac{d}{dt}[\sin t]\cdot t - \sin t \cdot \frac{d}{dt}[t]}{t^2}$$

$$= \frac{t\cos t - \sin t}{t^2}$$

$$\boxed{\vec{r}\,'(t) = \left\langle e^{-t} - te^{-t}, \frac{2t}{t^2+1}, \frac{t\cos t - \sin t}{t^2} \right\rangle}$$

Derivatives of vector-valued functions graphically:



Estimate slope of tangent line.

x-component of $\vec{r}\,'(t)$:

$$\frac{x(t+\Delta t) - x(t)}{\Delta t} \longrightarrow x'(t)$$

Same for y and z components

Def. $\vec{r}(t)$ is differentiable where its component functions are differentiable.

If $\vec{r}\,'(t) \neq 0$, then $\vec{r}\,'(t)$ is a tangent vector corresponding to $\vec{r}(t)$.

The unit tangent vector is $\boxed{\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}}$

Ex.2 Find the unit tangent vector at $t = -1$ for $\vec{r}(t) = \langle \ln(t+2), t^2, e^{t+1} \rangle$, $\underbrace{-2 < t}_{\text{Defined for these values of the parameter.}}$.

① $\vec{r}\,'(t) = \langle \frac{1}{t+2}, 2t, e^{t+1} \rangle$

② Compute general $\vec{T}(t)$ in terms of t,
OR evaluate $\vec{r}\,'(-1)$ and find specific $\vec{T}(-1)$.

$\vec{r}\,'(-1) = \langle \frac{1}{1}, -2, e^0 \rangle = \langle 1, -2, 1 \rangle$

$|\vec{r}\,'(-1)| = \sqrt{1+4+1} = \sqrt{6}$

③ $\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \boxed{\dfrac{\langle 1, -2, 1 \rangle}{\sqrt{6}}} = \langle \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}} \rangle$

Ex. 3 Find the line tangent to the curve
$\vec{r}(t) = \langle t^2 + 7, 4\cos t, 1 - \sin t\rangle$ at $t=0$.

To write the equation of a line, we need a point on the line and a direction. Since the line is tangent to the curve, the direction will be $\vec{r}\,'(0)$, and the position vector for a point on the curve <u>and</u> the tangent line is $\vec{r}(0)$.

① Find point on line.

$\vec{r}_0 = \vec{r}(0) = \langle 7, 4, 1\rangle$, so $P_0(7, 4, 1)$

② Find the direction.

$\vec{v} = \vec{r}\,'(t) = \langle 2t, -4\sin t, -\cos t\rangle$

$\vec{v} = \vec{r}\,'(0) = \langle 0, 0, -1\rangle$

③ Write vector or parametric equations for line.

$$\vec{r}_\ell = \vec{r}_0 + t\vec{v} = \boxed{\langle 7, 4, 1\rangle + t\langle 0, 0, -1\rangle = \vec{r}_\ell}$$

$$\boxed{\begin{aligned} x &= 7 \\ y &= 4 \\ z &= 1 - t \end{aligned}}$$

Thm 14.1 Derivative Rules

$\vec{u}, \vec{v}$ are vector-valued functions.
f is a scalar-valued function.
$\vec{c}$ is a constant vector

1. $\frac{d}{dt}[\vec{c}] = 0$
2. $\frac{d}{dt}[\vec{u}(t) + \vec{v}(t)] = \vec{u}'(t) + \vec{v}'(t)$
3. $\frac{d}{dt}[f(t)\,\vec{u}(t)] = f'(t)\,\vec{u}(t) + f(t)\,\vec{u}'(t)$
4. $\frac{d}{dt}[\vec{u}(f(t))] = f(t)\ \vec{u}'(f(t))$
5. $\frac{d}{dt}[\vec{u}(t) \cdot \vec{v}(t)] = \vec{u}'(t) \cdot \vec{v}(t) + \vec{u}(t) \cdot \vec{v}'(t)$
6. $\frac{d}{dt}[\vec{u}(t) \times \vec{v}(t)] = \vec{u}'(t) \times \vec{v}(t) + \vec{u}(t) \times \vec{v}'(t)$

Ex. 4 Compute $\frac{d}{dt}\left[\vec{u}\,(t^2+1)\right]$ where $\vec{u}(t) = \langle t^3 + t, \sin(t) \rangle$.

Note: If you write $\vec{u}(t^2+1)$ explicitly,
$\vec{u}(t^2+1) = \langle (t^2+1)^3 + t^2 + 1, \sin(t^2+1) \rangle$,
you will have to do multiple chain rules.

Using 4 from Thm 14.1:

$$\frac{d}{dt}\left[\vec{u}(t^2+1)\right] = \frac{d}{dt}\left[t^2+1\right]\vec{u}'(t^2+1)$$
$$= 2t\ \vec{u}'(t^2+1)$$
$$\left(\vec{u}'(t) = \langle 3t^2 + 1, \cos t \rangle\right)$$
$$= 2t \langle 3(t^2+1)^2 + 1, \cos(t^2+1) \rangle$$
$$= \boxed{\langle 6t(t^2+1)^2 + 2t,\ 2t\cos(t^2+1) \rangle}$$

Def. Indefinite Integration of $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$

$$\int \vec{r}(t)\,dt = \left\langle \int f(t)\,dt,\ \int g(t)\,dt,\ \int h(t)\,dt \right\rangle + \vec{C}$$

where $\vec{C} = \langle C_1, C_2, C_3 \rangle$ is a constant vector.

Definite Integration

$$\int_a^b \vec{r}(t)\,dt = \left\langle \int_a^b f(t)\,dt,\ \int_a^b g(t)\,dt,\ \int_a^b h(t)\,dt \right\rangle$$

Ex. 5 Evaluate $\int_0^2 e^t (t\hat{i} + 2\vec{j} - e^t\vec{k})\,dt$.

Need to compute

① $\int_0^2 te^t\,dt$

Integration by parts

$u = t \qquad dv = e^t dt$

$du = dt \qquad v = e^t$

$= uv\Big|_0^2 - \int_0^2 v\,du$

$= te^t\Big|_0^2 - \int_0^2 e^t\,dt$

$= te^t - e^t\Big|_0^2$

$= 2e^2 - e^2 - (0 - \underbrace{e^0}_{=1})$

$= 2e^2 - e^2 + 1$

$= e^2 + 1$

② $\int_0^2 2e^t\,dt$

$= 2e^t\Big|_0^2$

$= 2e^2 - 2\underbrace{e^0}_{=1}$

$= 2e^2 - 2$

③ $\int_0^2 -e^t e^t\,dt$

$= -\int_0^2 e^{2t}\,dt$

Substitution

$u = 2t$

$du = 2\,dt$

$= -\int_0^2 e^u \frac{du}{2}$

$= -\frac{1}{2}\int_0^2 e^u\,du$

$= -\frac{1}{2} e^u\Big|_0^2$

$= -\frac{1}{2} e^{2t}\Big|_0^2$

$= -\frac{1}{2}[e^4 - e^0]$

$= -\frac{1}{2}[e^4 - 1]$

$\int_0^2 e^t(t\vec{i} + 2\vec{j} - e^t\vec{k})\,dt$

$$\boxed{= (e^2+1)\vec{i} + (2e^2-2)\vec{j} - \frac{1}{2}(e^4-1)\vec{k}}$$

§ 14.3 Motion in Space

Def. If $\vec{r}(t)$ is a position function, then we can find:

s(t) is the arclength function we'll use later. →

velocity $\vec{v}(t) = \vec{r}\,'(t)$

speed $\frac{ds}{dt} = |\vec{v}(t)| = |\vec{r}\,'(t)|$

acceleration $\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t)$

where $t \geq 0$.

Given an acceleration $\vec{a}(t)$, we have

$$\vec{v}(t) = \int \vec{a}(t)\,dt$$

$$\frac{ds}{dt} = \left|\int \vec{a}(t)\,dt\right|$$

$$\vec{r}(t) = \int \vec{v}(t)\,dt$$

Note: These are indefinite integrals, so we will need to be given values to solve for the constant $\vec{C}$.

Ex. 6 Given position function $\vec{r}(t) = \langle t\sin t, t^2, t\cos t\rangle$, $t \geq 0$, find (a) velocity, (b) speed, and (c) acceleration.

(a) $\vec{v}(t) = \vec{r}\,'(t) = \left\langle \underbrace{\frac{d}{dt}[t\sin t]}_{\text{Product}}, \frac{d}{dt}[t^2], \underbrace{\frac{d}{dt}[t\cos t]}_{\text{Product}} \right\rangle$

$$= \boxed{\langle \sin t + t\cos t,\ 2t,\ \cos t - t\sin t\rangle}$$

(b) $\frac{ds}{dt} = |\vec{v}(t)| = \sqrt{(\sin t + t\cos t)^2 + (2t)^2 + (\cos t - t\sin t)^2}$

$$= \sqrt{(\sin^2 t + \cancel{2t\sin t\cos t} + t^2\cos^2 t) + (4t^2) + (\cos^2 t \cancel{- 2t\cos t\sin t} + t^2\sin^2 t)}$$

$$= \sqrt{\underbrace{\sin^2 t + \cos^2 t}_{=1} + t^2\underbrace{(\cos^2 t + \sin^2 t)}_{=1} + 4t^2}$$

$$= \sqrt{1 + t^2 + 4t^2}$$

$$= \boxed{\sqrt{1 + 5t^2}}$$

(c) $\vec{a}(t) = \vec{v}\,'(t) = \left\langle \frac{d}{dt}\left[\sin t + \underbrace{t\cos t}_{\text{Product}}\right], \frac{d}{dt}[2t], \frac{d}{dt}\left[\cos t - \underbrace{t\sin t}_{\text{Product}}\right] \right\rangle$

$= \langle \cos t + \cos t - t\sin t,\ 2,\ -\sin t - \sin t - t\cos t \rangle$

$= \boxed{\langle 2\cos t - t\sin t,\ 2,\ -2\sin t - t\cos t \rangle}$

Ex. 7 Let $\vec{a}(t) = t\,\vec{i} - t^3\,\vec{j} + 3t^5\,\vec{k}$, $\vec{r}(0) = \vec{i} - \vec{j}$, and $\vec{v}(0) = \vec{i}$.
Find (a) $\vec{v}(t)$ and (b) $\vec{r}(t)$.

(a) $\vec{v}(t) = \int \vec{a}(t)\,dt = \left(\int t\,dt\right)\vec{i} - \left(\int t^3\,dt\right)\vec{j} + \left(\int 3t^5\,dt\right)\vec{k}$

$= \left(\frac{1}{2}t^2 + C_1\right)\vec{i} - \left(\frac{1}{4}t^4 + C_2\right)\vec{j} + \left(\frac{3}{6}t^6 + C_3\right)\vec{k}$

OR: $\langle \frac{1}{2}t^2, -\frac{1}{4}t^4, \frac{1}{2}t^6 \rangle + \vec{C}$ where $\vec{C} = \langle C_1, C_2, C_3 \rangle$

Use $\vec{v}(0) = \vec{i}$ to find $\vec{C}$

$\vec{i} = \vec{v}(0) = \underbrace{(0 + C_1)}_{=1}\vec{i} - \underbrace{(0 + C_2)}_{=0}\vec{j} + \underbrace{(0 + C_3)}_{=0}\vec{k}$

$\Rightarrow C_1 = 1 \qquad \Rightarrow C_2 = 0 \qquad \Rightarrow C_3 = 0$

OR $\langle 1, 0, 0 \rangle = \langle 0, 0, 0 \rangle + \vec{C}$

$\langle 1, 0, 0 \rangle = \vec{C}$

Either way, $\boxed{\vec{v}(t) = \left(\frac{1}{2}t^2 + 1\right)\vec{i} - \frac{1}{4}t^4\,\vec{j} + \frac{1}{2}t^6\,\vec{k}}$

(b) $\vec{r}(t) = \int \vec{v}(t)\,dt$

$= \left(\int \left(\frac{1}{2}t^2 + 1\right)dt\right)\vec{i} - \left(\int \frac{1}{4}t^4\,dt\right)\vec{j} + \left(\int \frac{1}{2}t^6\,dt\right)\vec{k}$

$= \left(\frac{1}{6}t^3 + t + C_1\right)\vec{i} - \left(\frac{1}{20}t^5 + C_2\right)\vec{j} + \left(\frac{1}{14}t^7 + C_3\right)\vec{k}$

Use $\vec{r}(0) = \vec{i} - \vec{j}$ to find $\vec{C}$.

$\vec{i} - \vec{j} = \vec{r}(0)$

$= \underbrace{(C_1)}_{=1}\vec{i} - \underbrace{(C_2)}_{=1}\vec{j} + \underbrace{(C_3)}_{=0}\vec{k}$

$\Rightarrow C_1 = 1 \quad \Rightarrow C_2 = 1 \quad \Rightarrow C_3 = 0$

$$\boxed{\vec{r}(t) = \left(\frac{1}{6}t^3 + t + 1\right)\vec{i} - \left(\frac{1}{20}t^5 + 1\right)\vec{j} + \left(\frac{1}{14}t^7\right)\vec{k}}$$

Ex. 8 Let $\vec{r}(t) = \langle 6t, 3t^2, t^3 \rangle$. At what value(s) of t does the speed equal 18?

① First, find $\frac{ds}{dt} = |\vec{v}(t)| = |\vec{r}\,'(t)|$.

$$\vec{r}\,'(t) = \langle 6, 6t, 3t^2 \rangle$$

$$\begin{aligned} |\vec{r}\,'(t)| &= \sqrt{6^2 + (6t)^2 + (3t^2)^2} \\ &= \sqrt{36 + 36t^2 + 9t^4} \\ &= \sqrt{9(t^4 + 4t^2 + 4)} \\ &= \sqrt{9(t^2+2)^2} \\ &= 3(t^2+2) \end{aligned}$$

② Set $\frac{ds}{dt} = 18$ and solve for t.

$$\begin{aligned} 18 &= 3(t^2+2) \\ 6 &= t^2+2 \\ 4 &= t^2 \\ \pm 2 &= t \end{aligned}$$

$\boxed{\pm 2 = t}$

Ex. 9 Given $\vec{r}(t) = \langle \underbrace{t\sqrt{t}}_{=t^{3/2}}, 2t+1, t^2 - 8t \rangle$, at what t-value(s) does the minimum speed occur?

① Again, need to find $\frac{ds}{dt} = |\vec{r}\,'(t)|$.

$$\vec{r}\,'(t) = \langle \tfrac{3}{2}t^{1/2}, 2, 2t-8 \rangle$$

$$|\vec{r}\,'(t)| = \sqrt{\tfrac{9}{4}t + 4 + (4t^2 - 32t + 64)} = \sqrt{4t^2 - \tfrac{119}{4}t + 68}$$

② A min or max occurs when $\frac{d}{dt}\left[|\vec{r}\,'(t)|\right] = 0$.

Must have numerator $= 0$.

$$\frac{d}{dt}\left[|\vec{r}\,'(t)|\right] = \frac{8t - \frac{119}{4}}{2\left(4t^2 - \frac{119}{4}t + 68\right)^{1/2}} = 0$$

$$0 = 8t - \tfrac{119}{4}$$

$$\tfrac{119}{4} = 8t \quad \Rightarrow \quad \boxed{t = \tfrac{119}{32}}$$

Should use 1st or 2nd Derivative Test to check if min or max.