§ 14.3 Motion in Space (Part 2)

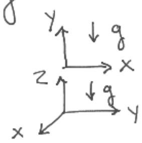
Recall: Position: $\vec{r}(t) = \int \vec{v}(t)\,dt$
Velocity: $\vec{v}(t) = \vec{r}\,'(t)$ and $v(t) = \int \vec{a}(t)\,dt$
Acceleration: $\vec{a}(t) = \vec{v}\,'(t)$

Speed is scalar. $\frac{ds}{dt} = |\vec{v}(t)|$, where $s(t)$ is arclength.

Def. The curve described by $\vec{r}(t)$ is the path or trajectory of the object.

If the only acceleration an object has is acceleration due to gravity, we have

2D: $\vec{a}(t) = \langle 0, g \rangle$
3D: $\vec{a}(t) = \langle 0, 0, g \rangle$



where $g = -9.8\ m/s^2$ or $g = -32.2\ ft/s^2$.

Pay close attention to units!

Suppose we have a projectile object launched from initial position $\vec{r}_0 = \langle x_0, y_0, z_0 \rangle$ with initial velocity $\vec{v}_0 = \langle a, b, c \rangle$ only acted upon by gravity.

Then, we have $\vec{a}(t) = \langle 0, 0, g \rangle$
$\vec{v}(t) = \int \vec{a}(t)\,dt = \langle C_1, C_2, gt + C_3 \rangle$

and we can use $\vec{V}_0$ to solve for $\vec{C}$.

Now, we have $\vec{v}(t) = \langle a, b, gt + c \rangle = \vec{v}_0 + \langle 0, 0, gt \rangle$

Next, we find $\vec{r}(t) = \int \vec{v}(t)\,dt$
$= \langle at + D_1, bt + D_2, \frac{1}{2}gt^2 + ct + D_3 \rangle$

and use $\vec{r}_0$ to solve for $\vec{D}$.

This yields $\vec{r}(t) = \langle at + x_0, bt + y_0, \frac{1}{2}gt^2 + ct + z_0 \rangle$
$= \vec{r}_0 + \vec{v}_0 t + \langle 0, 0, \frac{1}{2}gt^2 \rangle$

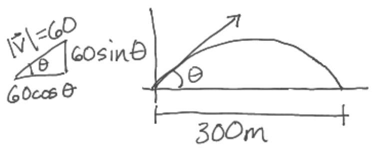
We usually assume the ground is $z=0$ (for 3D) or $y=0$ (for 2D).

Def. **Range**: the horizontal distance from where an object was launched to where it hits the ground

Time of flight: how long it takes for the projectile to hit the ground (when $t=0$ is the time it was launched)

Maximum height: In 2D, the maximum y-value. In 3D, the maximum z-value. In either case, the maximum height occurs when the vertical velocity is 0.

Ex. 1 A projectile is fired over horizontal ground from the origin with an initial speed of 60m/s. What firing angles produce a range of 300m?



$\vec{r}(0) = \langle 0, 0 \rangle$
$\vec{v}(0) = \langle 60\cos\theta, 60\sin\theta \rangle$
$\vec{a}(t) = \langle 0, -9.8 \rangle$

If the time of flight is T, we need $\vec{r}(T) = \langle 300, 0 \rangle$. First, we find $\vec{r}(t)$:

$$\begin{aligned}\vec{v}(t) &= \int \vec{a}(t)\,dt \\ &= \int \langle 0, -9.8 \rangle\,dt \\ &= \langle C_1, -9.8t + C_2 \rangle\end{aligned}$$

$$\begin{aligned}\vec{v}(0) &= \langle 60\cos\theta, 60\sin\theta \rangle \\ &= \langle C_1, C_2 \rangle\end{aligned}$$

$$\vec{v}(t) = \langle 60\cos\theta, -9.8t + 60\sin\theta \rangle$$

$$\begin{aligned}\vec{r}(t) &= \int \vec{v}(t)\,dt \\ &= \langle 60\cos\theta\, t + C_1, -4.9t^2 + 60\sin\theta\, t + C_2 \rangle\end{aligned}$$

$$\begin{aligned}\vec{r}(0) &= \langle 0, 0 \rangle \\ &= \langle C_1, C_2 \rangle\end{aligned}$$

$$\vec{r}(t) = \langle (60\cos\theta)t, -4.9t^2 + (60\sin\theta)t \rangle$$

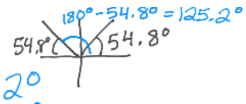
Now, we need to find the flight time T. This will be the time when the y-component of $\vec{r}(t)$ is 0.

$$\begin{aligned}-4.9T^2 + (60\sin\theta)T &= 0 \\ T\underbrace{(-4.9T + 60\sin\theta)}_{T = \frac{60\sin\theta}{4.9}} &= 0\end{aligned}$$

Finally, we want to set the x-component of $\vec{r}(T)$ to be 300.

$$\begin{aligned}(60\cos\theta)T &= 300 \qquad \text{Use } T = \frac{60\sin\theta}{4.9}. \\ (60\cos\theta)\left(\frac{60\sin\theta}{4.9}\right) &= 300 \\ 3600\cos\theta\sin\theta &= 300(4.9) \qquad \text{Recall: } 2\cos\theta\sin\theta = \sin(2\theta) \\ 1800\sin(2\theta) &= 1470 \\ \sin(2\theta) &= 49/60\end{aligned}$$

$$2\theta = \arcsin(49/60)$$
$$2\theta = 54.8^\circ \quad \text{or} \quad 2\theta = 125.2^\circ$$
$$\theta = 27.4^\circ \qquad\qquad \theta = 62.6^\circ$$



<u>Ans</u>: $\theta = 27.4^\circ$ or $\theta = 62.6^\circ$

<u>Ex. 2</u> The acceleration due to gravity on the moon is one-sixth its value on Earth. Compare the time of flight, range, and maximum height of a projectile on the moon with the corresponding values on Earth.

Since we're comparing values on the moon and on Earth, we can make assumptions on the initial conditions as long as we use the same assumptions for both locations.

<u>Assume</u>: 2D with $\vec{v}(0) = \langle a, b\rangle$ and $\vec{r}(0) = \langle 0, 0\rangle$.

<u>On Earth</u>:
$$\vec{a}(t) = \langle 0, g\rangle$$
$$\vec{v}(t) = \langle a, b + gt\rangle$$
$$\vec{r}(t) = \langle at, bt + \tfrac{1}{2}gt^2\rangle$$

<u>Time of flight T</u>: vertical position $= 0$
$$bT + \tfrac{1}{2}gT^2 = 0$$
$$\underbrace{T}_{T=0}\underbrace{\left(b + \tfrac{1}{2}gT\right)}_{T = -\frac{2b}{g}} = 0$$

<u>Range</u>: horizontal position at T
$$aT = -\frac{2ab}{g}$$

<u>Max height</u>: achieved when vertical velocity $= 0$
$$b + gt = 0$$
$$t = -\frac{b}{g}$$
$$b\left(-\frac{b}{g}\right) + \frac{1}{2}g\left(-\frac{b}{g}\right)^2$$
$$= -\frac{b^2}{g} + \frac{1}{2}\frac{b^2}{g} = -\frac{1}{2}\frac{b^2}{g}$$

<u>On Moon</u>: Replace g with $g/6$

<u>Time of flight</u>: $T = -\dfrac{2b}{(g/6)} = -\underline{6}\left(\dfrac{2b}{g}\right)$

<u>Range</u>: $-\dfrac{2ab}{(g/6)} = -\underline{6}\left(\dfrac{2ab}{g}\right)$

<u>Max height</u>: $-\dfrac{1}{2}\dfrac{b^2}{(g/6)} = -\underline{6}\left(\dfrac{b^2}{2g}\right)$

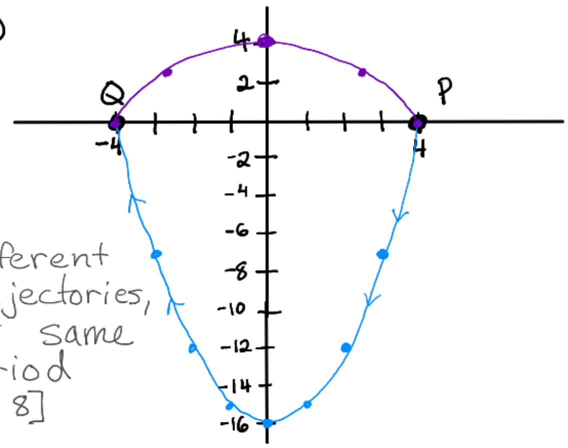
<u>Ans</u>: On the moon, all three values are 6 times what they are on Earth.

Ex.3 Two people travel from $P(4,0)$ to $Q(-4,0)$ along the paths given by

$$\vec{r}(t) = \langle 4\cos\tfrac{\pi t}{8},\ 4\sin\tfrac{\pi t}{8} \rangle$$
$$\vec{R}(t) = \langle 4 - t,\ (4-t)^2 - 16 \rangle$$

(a) Graph both paths between P and Q
(b) Graph the speeds of both people.
(c) Who arrives at Q first?

(a)



Note: different trajectories, but same period $[0, 8]$

t	$\vec{R}(t)$	$\vec{r}(t)$
0	$\langle 4, 0 \rangle$	$\langle 4, 0 \rangle$
1	$\langle 3, -7 \rangle$	
2	$\langle 2, -12 \rangle$	$\langle 2\sqrt{2}, 2\sqrt{2} \rangle$
3	$\langle 1, -15 \rangle$	
4	$\langle 0, -16 \rangle$	$\langle 0, 4 \rangle$
5	$\langle 1, -15 \rangle$	
6	$\langle 2, -12 \rangle$	$\langle -2\sqrt{2}, 2\sqrt{2} \rangle$
7	$\langle 3, -7 \rangle$	
8	$\langle 4, 0 \rangle$	$\langle 4, 0 \rangle$

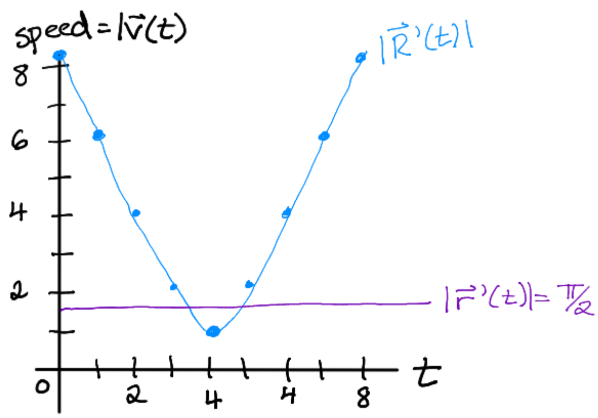
(b) $\vec{r}\,'(t) = \langle -\frac{\pi}{2}\sin(\frac{\pi t}{8}),\ \frac{\pi}{2}\cos(\frac{\pi t}{8}) \rangle$

$|\vec{r}\,'(t)| = \sqrt{\frac{\pi^2}{4}\left(\sin^2(\frac{\pi t}{8}) + \cos^2(\frac{\pi t}{8})\right)} = \frac{\pi}{2}$

$\vec{R}\,'(t) = \langle -1, -2(4-t) \rangle$

$|\vec{R}\,'(t)| = \sqrt{1 + 4(4-t)^2} = \sqrt{4t^2 - 32t + 65}$

t	$\lvert\vec{R}\,'(t)\rvert$	$\lvert\vec{r}\,'(t)\rvert$
0	$\sqrt{65}$	$\pi/2$
1	$\sqrt{37}$	
2	$\sqrt{17}$	
3	$\sqrt{5}$	
4	1	
5	$\sqrt{5}$	
6	$\sqrt{17}$	
7	$\sqrt{37}$	
8	$\sqrt{65}$	



(c) They both arrive when $t = 8$.

§14.4 Length of Curves

Def. The arc length of the curve $\vec{r}(t)$ traversed once for $a \le t \le b$ is $\boxed{L = \int_a^b |\vec{r}\,'(t)|\,dt}$

Ex.4 Find the length of the curve $\vec{r}(t) = \langle 2\cos t, 2\sin t, \ln t - \frac{1}{2}t^2 \rangle$ for $1 \le t \le 2$.

$\vec{r}\,'(t) = \langle -2\sin t, 2\cos t, \frac{1}{t} - t \rangle$

$|\vec{r}\,'(t)| = \sqrt{4(\sin^2 t + \cos^2 t) + (\frac{1}{t^2} - 2 + t^2)}$

$= \sqrt{t^2 + 2 + \frac{1}{t^2}}$

$= \sqrt{(t + \frac{1}{t})^2}$

$= (t + \frac{1}{t})$

$L = \int_1^2 (t + \frac{1}{t})\,dt$

$= \frac{1}{2}t^2 + \ln(t) \Big|_1^2$

$= (2 + \ln(2)) - (\frac{1}{2} + 0)$

$= \boxed{\frac{3}{2} + \ln(2)}$

Def. The arc length function, $s(t)$, gives the length of the curve traced by $\vec{r}(t)$ (starting at $t=a$) for $t \ge a$. $\boxed{s(t) = \int_a^t |\vec{r}\,'(\tau)|\,d\tau}$

Recall: Speed is $\frac{ds}{dt} = |\vec{r}\,'(t)|$ where $s(t)$ is arc length.

We can parameterize a curve with respect to arc length by finding $s(t)$, then solving for t in terms of s ($s(t) \to t(s)$). Finally, plug $t(s)$ into $\vec{r}(t)$ to get $\vec{r}(s) = \vec{r}(t(s))$.

If a curve $\vec{r}(t)$ uses arc length as a parameter, then $|\vec{r}\,'(t)| = 1$.
Why? Because $\frac{ds}{dt} = 1 = \frac{dt}{dt}$.

Ex.5 Determine if $\vec{r}(t) = \langle \cos t^2, \sin t^2 \rangle$, $1 \le t \le \sqrt{\pi}$ uses arc length as a parameter. If not, reparameterize with arc length.

$|\vec{r}\,'(t)| = |\langle -2t\sin(t^2), 2t\cos(t^2) \rangle| = \sqrt{4t^2(\sin^2(t^2) + \cos^2(t^2))}$

$= 2t \ne 1$, so arc length is not the parameter.

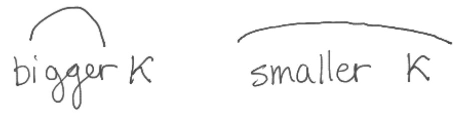
$s(t) = \int_1^t 2\tau\,d\tau = \tau^2 \Big|_1^t = t^2 - 1 \Rightarrow t = \sqrt{s+1}$ $\quad t=1: s=0$; $t=\sqrt{\pi}: s=\pi-1$

$\vec{r}(s) = \vec{r}(t(s)) = \vec{r}(\sqrt{s+1}) = \boxed{\langle \cos(s+1), \sin(s+1) \rangle, 0 \le s \le \pi - 1}$

§14.5 Curvature and Normal Vectors

Recall: The unit tangent vector of $\vec{r}(t)$ is $\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$.

Def. Curvature, $\kappa(t)$ (kappa), measures how fast a curve turns at a point:



$$\boxed{\kappa(t) = \frac{|\vec{T}\,'(t)|}{|\vec{r}\,'(t)|}}$$

If $\vec{r}(s)$ and $\vec{T}(s)$ are parameterized by arc length,

$$\boxed{\kappa(s) = \frac{d\vec{T}}{ds}}$$

Def. Unit normal vector $\vec{N}(t)$ determines the direction in which a curve turns.

$$\boxed{\vec{N}(t) = \frac{\vec{T}\,'(t)}{|\vec{T}\,'(t)|}}$$

Ex. 6 Show $\vec{N}(t)$ and $\vec{T}(t)$ are orthogonal at all points of a smooth curve.

$|\vec{T}(t)| = 1$ (constant), so $|\vec{T}(t)|^2 = \vec{T}(t)\cdot\vec{T}(t) = 1$

Differentiate: $\dfrac{d}{dt}[1] = \dfrac{d}{dt}[\vec{T}(t)\cdot\vec{T}(t)]$

$$0 = \vec{T}\,'(t)\cdot\vec{T}(t) + \vec{T}(t)\cdot\vec{T}\,'(t)$$
$$0 = 2\vec{T}\,'(t)\cdot\vec{T}(t)$$
$$0 = \vec{T}\,'(t)\cdot\vec{T}(t)$$
$$\Rightarrow \quad 0 = \left(|\vec{T}\,'(t)|\vec{N}(t)\right)\cdot\vec{T}(t)$$
$$0 = \vec{N}(t)\cdot\vec{T}(t) \quad \checkmark$$

Ex. 7 Find the unit tangent vector and curvature for $\vec{r}(t) = \langle t, \ln(\cos t)\rangle$.

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \frac{\langle 1, \overbrace{\frac{-\sin t}{\cos t}}^{=-\tan t}\rangle}{\sqrt{\underbrace{1+\tan^2 t}_{=\sec^2 t}}} = \frac{\langle 1, -\tan t\rangle}{\sec t} = \boxed{\langle \cos t, -\sin t\rangle}$$

$$\frac{-\frac{\sin t}{\cos t}}{\frac{1}{\cos t}} = -\sin t$$

$$\kappa(t) = \frac{|\vec{T}\,'(t)|}{|\vec{r}\,'(t)|} = \frac{\sqrt{\cos^2 t + \sin^2 t}}{\sec t} = \frac{1}{\sec t} = \boxed{\cos t}$$