Ch 15: Functions of Several Variables

§ 15.1 Graphs and Level Curves

A function of several variables has several inputs (a point) and a single output.

We will mostly deal with a function of 2 variables (x,y) with output z.
We write this explicitly as $z = f(x,y)$
and implicitly as $0 = F(x,y,z)$.

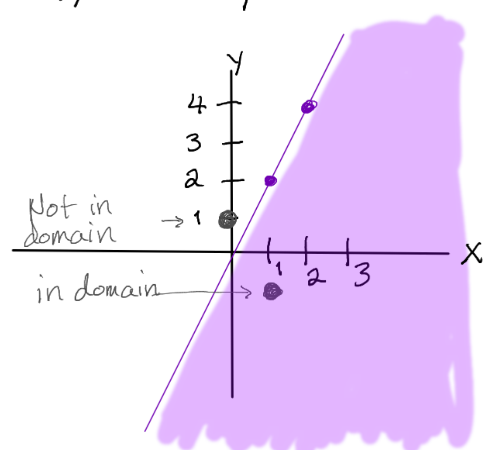
The domain will be a set of points (x,y) in $\mathbb{R}^2$ $(= \mathbb{R} \times \mathbb{R})$ meaning in the xy-plane.
We focus on the same domain issues for several variables that we do for single variables: denominators, even roots, and logs.

The range will be the possible z-values for the domain.

Ex. 1 Find and sketch the domain of $f(x,y) = \sqrt{2x - y}$.

Need $2x - y \geq 0$
$2x \geq y$

Write: $\boxed{\{(x,y) : 2x \geq y\}}$

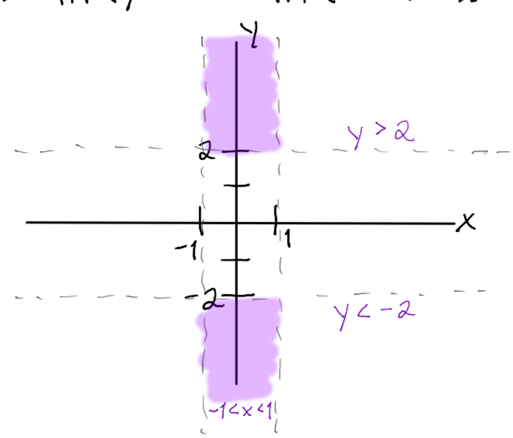


Graph the line $y = 2x$.
Since values on the line are in the domain, we should have a solid (not dashed) line. Next, determine if we shade above or below the line. To do this, we can plug in $(0,1)$ above and $(1,-1)$ below to check $2x \geq y$.
$(0,1)$: $0 \geq 1$? No – not in domain
$(1,-1)$: $2 \geq -1$? Yes – in domain

Ex. 2 Find and sketch the domain of $f(x,y) = \ln(y^2 - 4) - \ln(1 - x^2)$.
What is the range of f?

Need: $1 - x^2 > 0$ and $y^2 - 4 > 0$
$1 > x^2$ $\quad$ $y^2 > 4$
$\Rightarrow -1 < x < 1$ $\quad$ $\Rightarrow y < -2$ or $y > 2$

Write: $\boxed{\{(x,y) : -1 < x < 1 \text{ and } y < -2 \text{ or } y > 2\}}$



Range: For $-1 < x < 1$, $0 < 1 - x^2 < 1$, so $-\infty < \ln(1-x^2) < 0$
For $y < -2$ or $y > 2$, $0 < y^2 - 4 < \infty$, so $-\infty < \ln(y^2 - 4) < \infty$

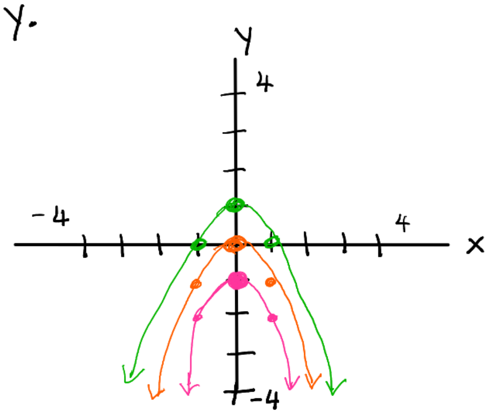
We can have infinitely large (positive or negative) first term minus an infinitely large negative second term. Then the range is $\boxed{\mathbb{R}}$. (All real numbers.)

Def. Level curves (or contour lines) of a function are curves drawn in the xy-plane for a constant $z = k$: $f(x,y) = k$.
Drawing multiple level curves on a single graph is a contour map.
(Basically, these are traces.)

Ex. 3 Draw a contour map of $f(x,y) = x^2 + y$.

$z = -1$: $x^2 + y = -1 \Rightarrow y = -1 - x^2$
$z = 0$: $x^2 + y = 0 \Rightarrow y = -x^2$
$z = 1$: $x^2 + y = 1 \Rightarrow y = 1 - x^2$

Range of f: $(-\infty, \infty) = \mathbb{R}$
Domain of f: $\mathbb{R}^2$



§15.2 Limits and Continuity

Def. The function f has the limit L as $P(x,y)$ approaches $P_0(a,b)$

i.e. $L = \lim_{(x,y)\to(a,b)} f(x,y)$

if $f(x,y)$ gets arbitrarily close to L as (x,y) gets arbitrarily close to (a,b).

To evaluate the limit of f as $(x,y) \to (a,b)$:

1. If (a,b) is in the domain of f, the limit is $f(a,b)$.
2. If (a,b) is not in the domain, try to simplify f and evaluate.
3. If ① and ② don't work, check the limit along different curves. Meaning, see if the limits are the same when $x=0$, $y=0$, $\underset{(m\neq 0)}{y=mx}$, $\underset{(m\neq 0)}{y=mx^2}$, $x=my^2$, etc.

It doesn't matter how we approach (a,b) as long as the points (x,y) get closer to (a,b).

Def. A function f is continuous at (a,b) if $\boxed{\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)}$.

Note: This implies that f is defined at (a,b) and the limit exists.

Ex.4 Find $\displaystyle \lim_{(x,y)\to(1,0)} \frac{2x+3y}{1+2x+y} = \frac{2(1)+3(0)}{1+2(1)+0} = \boxed{\frac{2}{3}}$

Ex.5 Find $\displaystyle \lim_{(x,y)\to(1,1)} \frac{x^2-y^2}{4x-4y} = \frac{0}{0} \to (1,1)$ is not in the domain.

Simplify: $\displaystyle \frac{(x+y)\cancel{(x-y)}}{4\cancel{(x-y)}} = \frac{x+y}{4}$

Now, $\displaystyle \lim_{(x,y)\to(1,1)} \frac{x+y}{4} = \frac{2}{4} = \boxed{\frac{1}{2}}$

Ex.6 Find $\displaystyle \lim_{(x,y)\to(0,0)} \frac{x^4-y^4}{x^4+y^4} = \frac{0}{0}$, $(0,0)$ not in domain.

Can't simplify.

Approach $(0,0)$ along y-axis $(x=0)$: $\displaystyle \lim_{(0,y)\to(0,0)} \frac{0-y^4}{0+y^4} = \lim_{(0,y)\to(0,0)} -1 = -1$

Approach $(0,0)$ along x-axis $(y=0)$: $\displaystyle \lim_{(x,0)\to(0,0)} \frac{x^4-0}{x^4+0} = 1$

These values are different, so the limit $\boxed{\text{does not exist}}$.

Ex.7 Find $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4} = \frac{0}{0} \to (0,0)$ not in domain.

Can't simplify.

$x=0$: $\lim_{(0,y)\to(0,0)} \frac{0}{0+y^4} = \lim_{(0,y)\to(0,0)} (0) = 0$

$y=0$: $\lim_{(x,0)\to(0,0)} \frac{0}{x^2+0} = 0$

Same. Check more curves

$y=mx$: $\lim_{(x,mx)\to(0,0)} \frac{x(mx)^2}{x^2+(mx)^4} = \lim_{x\to 0} \frac{m^2x^3}{x^2+m^4x^4} = \lim_{x\to 0} \frac{m^2x}{1+m^4x^2} = 0$

$y=mx^2$: $\lim_{(x,mx^2)\to(0,0)} \frac{x(mx^2)^2}{x^2+(mx^2)^4} = \lim_{x\to 0} \frac{m^2x^5}{x^2+m^4x^8} = \lim_{x\to 0} \frac{m^2x^3}{1+m^4x^6} = 0$

$x=my^2$: $\lim_{(my^2,y)\to(0,0)} \frac{(my^2)y^2}{(my^2)^2+y^4} = \lim_{y\to 0} \frac{my^4}{m^2y^4+y^4} = \lim_{y\to 0} \frac{m}{m^2+1} = \frac{m}{m^2+1}$

Since $m\neq 0$, this last value is different from 0, so the $\boxed{\text{limit does not exist}}$.

Ex.8 Find $\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1} = \frac{0}{0} \to (0,0)$ not in domain.

To simplify, we rationalize the denominator.

$$\frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1} \cdot \frac{\sqrt{x^2+y^2+1}+1}{\sqrt{x^2+y^2+1}+1} = \frac{(x^2+y^2)(\sqrt{x^2+y^2+1}+1)}{(\sqrt{x^2+y^2+1})^2-(1)^2}$$

$$= \frac{\cancel{(x^2+y^2)}(\sqrt{x^2+y^2+1}+1)}{\cancel{x^2+y^2}\cancel{+1}\cancel{-1}}$$

$$= \sqrt{x^2+y^2+1}+1$$

as $(x,y)\to(0,0)$ $\to \sqrt{1}+1 = \boxed{2}$

Ex. 9 The temperature $T(x,y)$ at the center of a cake depends on the thickness y and diameter x of the cake. Values for $T(x,y)$ are given in the table.

$y \downarrow \backslash x \rightarrow$	6	7	8	9
1.5	210	205	200	195
2	200	195	190	185
2.5	195	18	175	165
3	175	165	155	145

(a) What is $T(7,2)$, and what does it mean?

(b) What does $T(7,y)$ mean?

(c) What is $T(x,3)$? Describe its behavior.

(a) $T(7,2) = 195$; it means that a cake of thickness 2 and diameter 7 has a temperature of 195 at the center.

(b) $T(7,y)$ is the temperature at the center of a cake of radius 7 with thickness y.

(c) $T(x,3) = 175 - 10(x-6)$
It is linear. As x increases, $T(x,3)$ decreases.

Match each function (1 – 6) with its graph (A – F) and its contour map (I – VI).

↳ Closer contour lines means a steeper surface.

1. $f(x,y) = \sin(xy)$

2. $f(x,y) = \sin(x-y)$

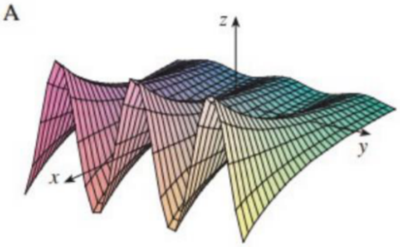
3. $f(x,y) = (1-x^2)(1-y^2)$

4. $f(x,y) = e^x \cos(y)$

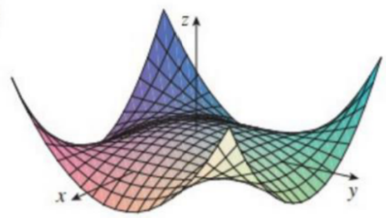
5. $f(x,y) = \sin(x) - \sin(y)$

6. $f(x,y) = \frac{x-y}{1+x^2+y^2}$

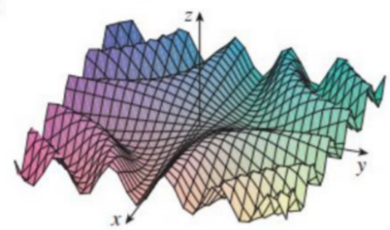
A



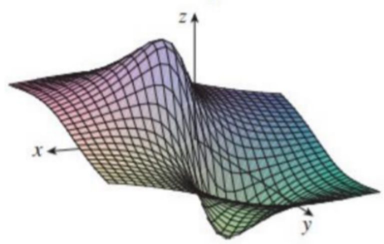
B



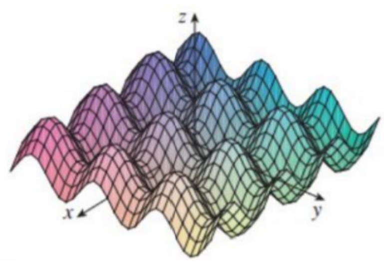
C



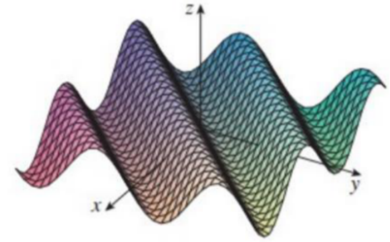
D



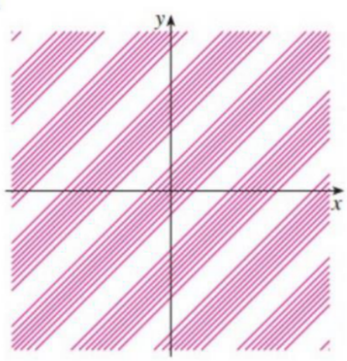
E



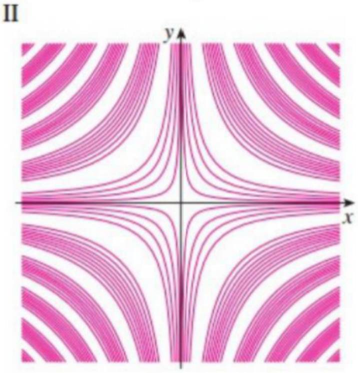
F



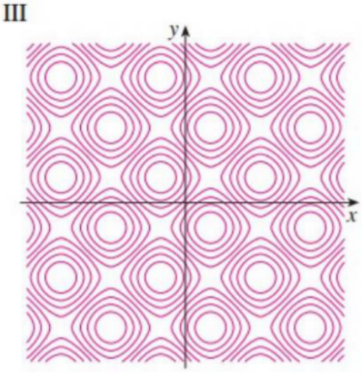
I



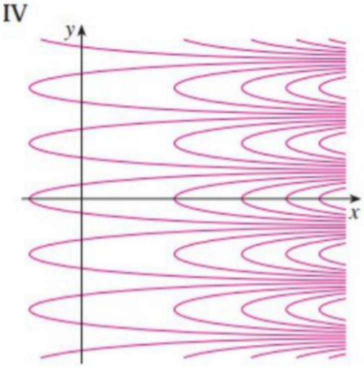
II



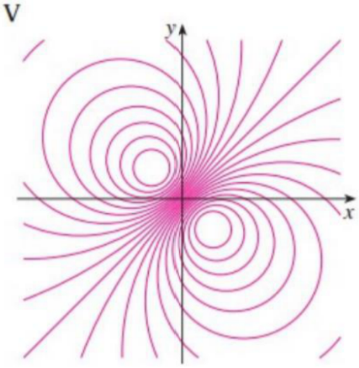
III



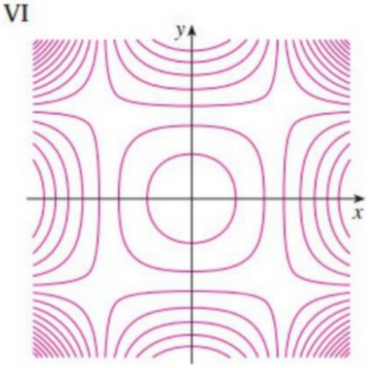
IV



V



VI

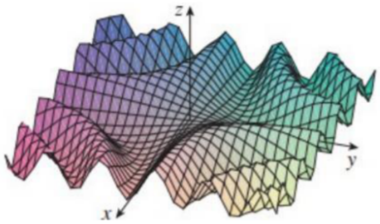


1. $f(x,y) = \sin(xy)$ $\rightarrow$ $f(x,y) = 0$

$\sin(xy) = 0$

$\Rightarrow$ $xy = \pi n$ (n is any integer)

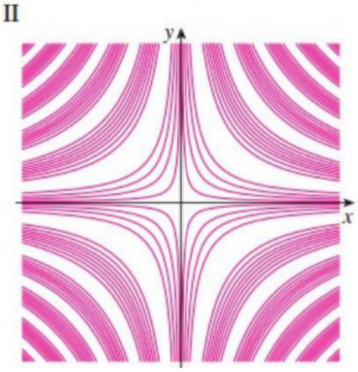
C



Easiest to look at $xy = 0 \Rightarrow x = 0$ or $y = 0$.

This means that on the x- and y-axes, $z = 0$. You can see that only Graph C has this property.

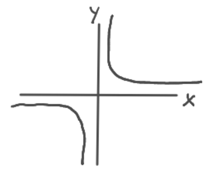
II



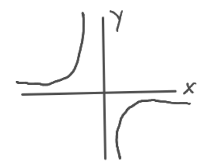
Thinking about contour lines, we have $k = \sin(xy)$,

$\Rightarrow xy = \arcsin(k)$ (a number)

$\Rightarrow y = \dfrac{\arcsin(k)}{x}$ whose graph is similar to



if $\arcsin(k) > 0$



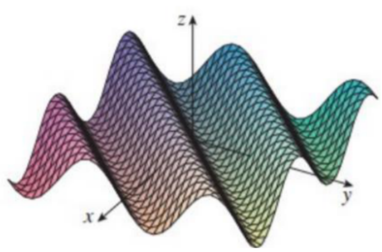
if $\arcsin(k) < 0$

Contour map II has these level curves.

2. $f(x,y) = \sin(x-y)$ $\rightarrow$ $f(x,y) = 0$ when $\sin(x-y) = 0$

$\Rightarrow$ $x - y = \pi n$

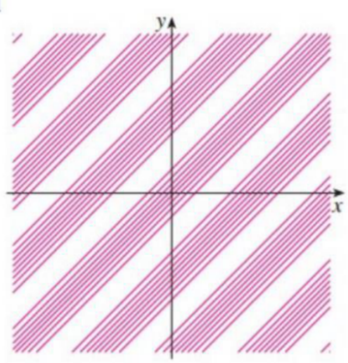
F



Again, we look at $x - y = 0$, so $x = y$.
This means that along the line $y = x$, $z = 0$.

Only Graph F has this property.

I



The contour lines have the form $k = \sin(x-y)$,

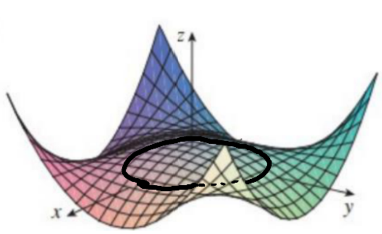
so $x - y = \arcsin(k)$

$y = x - \underbrace{\arcsin(k)}_{\#}$.

This means that the contour map consists of lines with slope $= 1$ and y-intercepts at $y = -\arcsin(k)$, so we have contour map F.

3. $f(x,y) = (1-x^2)(1-y^2)$ This function is more complicated.
Once again, we see where $z = 0$.
This happens when $x = \pm 1$ or $y = \pm 1$.
Let's make a sign chart:

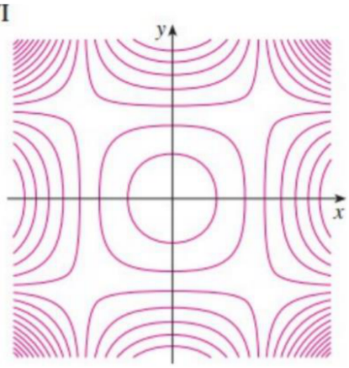
B



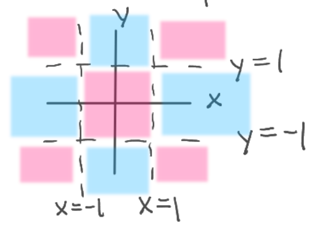
$f(x,y)$	$1-x^2$	$1-y^2$	x	y
+	+	+	$\lvert x\rvert < 1$	$\lvert y\rvert < 1$
+	−	−	$\lvert x\rvert > 1$	$\lvert y\rvert > 1$
−	+	−	$\lvert x\rvert < 1$	$\lvert y\rvert > 1$
−	−	+	$\lvert x\rvert > 1$	$\lvert y\rvert < 1$

Looking at B, we can see that inside the circle where $|x|<1$ and $|y|<1$, $f(x,y)$ is positive. No other surface has this property.

VI



Looking at the sign chart, the xy-plane can be separated like so:

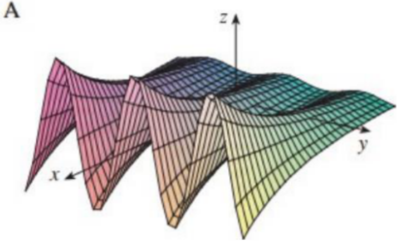


When we fix z, the curves we get will <u>all</u> be in the pink or <u>all</u> be in the blue. The only contour map separated this way is $\underline{\text{VI}}$

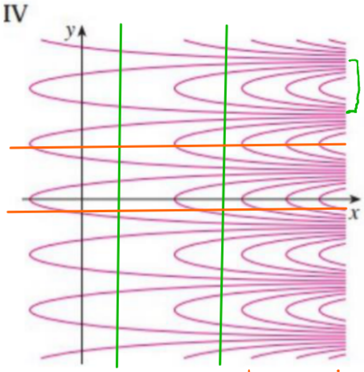
4. $f(x,y) = e^x \cos(y)$

If we fix $x = x_0$, the traces are of the form $z = \underbrace{e^{x_0}}_{\#} \cdot \cos(y)$, so traces parallel to the yz-plane are cosine functions. If we fix $y = y_0$, the traces are of the form $z = e^x \underbrace{\cos(y_0)}_{\#}$, so traces parallel to the xz-plane are exponential if $\cos(y_0) \neq 0$. If $\cos(y_0) = 0$, $z = 0$. This gives Graph A.

A



IV



periodic

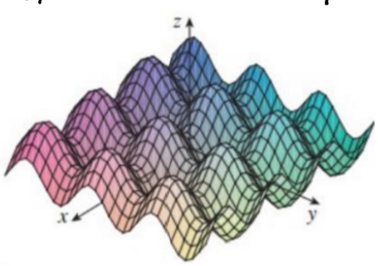
lines getting closer

For the contour map, we have level curves of the form $k = e^x \cos(y)$. Solving for y is not very helpful here. Instead, think again about constant x- or y-values. Along <u>constant x-values</u>, we should have periodic level curves. Along <u>constant y-values</u>, we should have exponentially increasing level curves.

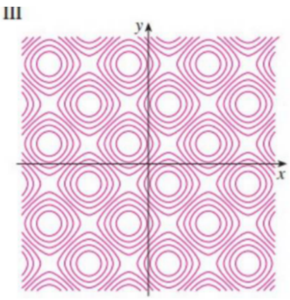
5. $f(x,y) = \sin(x) - \sin(y)$

For $x = x_0$, we have $z = \# - \sin(y)$, so we have shifted sine curves parallel to the yz-axis. For $y = y_0$, we have $z = \sin(x) - \#$, so again, shifted sine curves parallel to the xz-axis. This gives E.

E



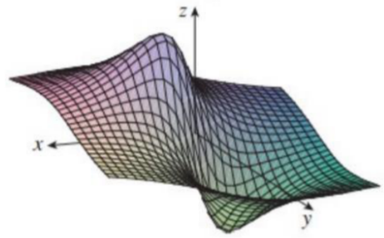
III



The level curves should be periodic along both horizontal and vertical lines. This gives $\underline{\text{III}}$.

6. $f(x,y) = \dfrac{x-y}{1+x^2+y^2}$

D

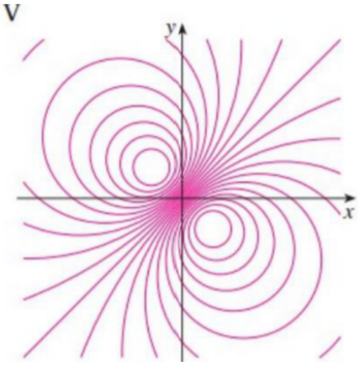


$$x = y \Rightarrow x - y = 0 \Rightarrow z = 0$$
$$x > y \Rightarrow x - y > 0 \Rightarrow z > 0$$
$$x < y \Rightarrow x - y < 0 \Rightarrow z < 0$$

($1+x^2+y^2$ always positive.)

Looking at the graphs, we see that this is true for Graph D.

V



For level curves, $k = \dfrac{x-y}{1+x^2+y^2}$, $k>0$

$\Rightarrow k(1+x^2+y^2) = x-y$

$\Rightarrow k\left(x^2 - \frac{x}{k} + \left(\frac{1}{2k}\right)^2\right) + k\left(y^2 + \frac{y}{k} + \left(\frac{1}{2k}\right)^2\right) = -k + 2k\left(\frac{1}{2k}\right)^2$

$\Rightarrow \left(x - \frac{1}{2k}\right)^2 + \left(y + \frac{1}{2k}\right)^2 = k$

$\Rightarrow$ The level curves are circles centered at $\left(\underset{\substack{>0\\+x}}{\frac{1}{2k}}, \underset{\substack{<0\\-y}}{-\frac{1}{2k}}\right)$ with radius $\sqrt{k}$.

Note: If $k<0$, we move all values to the other side and get

$\left(x - \frac{1}{2k}\right)^2 + \left(y + \frac{1}{2k}\right)^2 = -k$.

Circles centered at $\left(\underset{\substack{<0\\-x}}{\frac{1}{2k}}, \underset{\substack{>0\\+y}}{-\frac{1}{2k}}\right)$ with radius $\sqrt{-k}$.