§ 15.3 Partial Derivatives

Def. If $z = f(z, y)$ the first partial derivative with respect to x is $f_x(x,y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial z}{\partial x}$ and is computed at point (a,b)

as $f_x(a,b) = \frac{\partial f}{\partial x}\Big|_{(a,b)} = \lim_{h \to 0} \frac{f(a+h, b) - f(a,b)}{h}$.

In practice, treat y as a constant and take the derivative with respect to x.

We define the partial with respect to y similarly.

For second partials,

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

$$f_{\underline{xy}} = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right] = \frac{\partial^2 f}{\partial y\, \partial x} = \frac{\partial^2 z}{\partial y\, \partial x}$$

Compute from left to right.

$$f_{\underline{yx}} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right] = \frac{\partial^2 f}{\partial x\, \partial y} = \frac{\partial^2 z}{\partial x\, \partial y}$$

$$f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}$$

Note: For most functions we'll encounter, $f_{xy} = f_{yx}$, but this is not always true.

Ex. 1 Find all second partials of $f(x,y) = 4x^2y + x^3y^4$.

$$\begin{aligned} f_x &= \tfrac{\partial}{\partial x}[4x^2y] + \tfrac{\partial}{\partial x}[x^3y^4] \\ &= 4y\, \tfrac{\partial}{\partial x}[x^2] + y^4 \tfrac{\partial}{\partial x}[x^3] \\ &= 4y\,(2x) + y^4[3x^2] \\ &= 8xy + 3x^2y^4 \end{aligned}$$

Factor out constants. $((cf(x))' = cf'(x))$

$$f_{xx} = \tfrac{\partial}{\partial x}[f_x] = 8y\, \tfrac{\partial}{\partial x}[x] + 3y^4 \tfrac{\partial}{\partial x}[x^2] = \boxed{8y + 6xy^4} = f_{xx}$$

$$f_{xy} = \tfrac{\partial}{\partial y}[f_x] = 8x\, \tfrac{\partial}{\partial y}[y] + 3x^2 \tfrac{\partial}{\partial y}[y^4] = \boxed{8x + 12x^2y^3} = f_{xy}$$

$$f_y = 4x^2 \tfrac{\partial}{\partial y}[y] + x^3 \tfrac{\partial}{\partial y}[y^4] = 4x^2 + 4x^3y^3$$

$$f_{yx} = 4\, \tfrac{\partial}{\partial x}[x^2] + 4y^3 \tfrac{\partial}{\partial x}[x^3] = \boxed{8x + 12x^2y^3} = f_{yz}$$

$$f_{yy} = 4x^2 \overset{=0}{\cancel{\tfrac{\partial}{\partial y}[1]}} + 4x^3 \tfrac{\partial}{\partial y}[y^3] = \boxed{12x^3y^2} = f_{yy}$$

Ex. 2 Find all second partials of $f(u,v) = \ln(u^2+v) + e^{u^2-v^2}$.

$$f_u = \frac{2u}{u^2+v} + 2ue^{u^2-v^2}$$

$$f_{uu} = \frac{2(u^2+v) - 2u(2u)}{(u^2+v)^2} + 2e^{u^2-v^2} + 4u^2e^{u^2-v^2} = \boxed{\frac{2v-2u^2}{(u^2+v)^2} + (2+4u^2)e^{u^2-v^2}}$$

$$\boxed{f_{uv} = \frac{-2u}{(u^2+v)^2} - 4uve^{u^2-v^2}}$$

$$f_v = \frac{1}{u^2+v} - 2ve^{u^2-v^2}$$

$$\boxed{f_{vu} = \frac{-2u}{(u^2+v)^2} - 4uve^{u^2-v^2}}$$

$$f_{vv} = \frac{-1}{(u^2+v)^2} - 2e^{u^2-v^2} + 4v^2e^{u^2-v^2} = \boxed{\frac{-1}{(u^2+v)^2} + (4v^2-2)e^{u^2-v^2}}$$

Ex. 3 Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ for $f(x,y) = \int_{x^2}^{y-x} g(t)\,dt$ (g continuous for all t).

Recall: (Fundamental Theorem of Calculus)

$$f(x) = \int_{h_1(x)}^{h_2(x)} g(t)\,dt \qquad (g \text{ continuous for all } t)$$

$$\int g(t)\,dt = G(t)$$

$$= G(t)\Big|_{h_1(x)}^{h_2(x)}$$

$$= G(h_2(x)) - G(h_1(x))$$

$$\Rightarrow f'(x) = h_2'(x)\cdot g(h_2(x)) - h_1'(x)\cdot g(h_1(x)) \quad \text{(chain rule)}$$

Here: $f(x,y) = \int_{x^2}^{y-x} g(t)\,dt$

$$= G(t)\Big|_{x^2}^{y-x}$$

$$= G(y-x) - G(x^2)$$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}[y-x]\,g(y-x) - \frac{\partial}{\partial x}[x^2]\,g(x^2)$$

$$\boxed{\frac{\partial f}{\partial x} = -g(y-x) - 2x\,g(x^2)}$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}[y-x]\,g(y-x) - \overbrace{\cancel{\frac{\partial}{\partial y}[x^2]}}^{=0}\,g(x^2)$$

$$\boxed{\frac{\partial f}{\partial y} = g(y-x)}$$

§15.4 The Chain Rule

We could write z as a function of t by substituting. t is the only independant variable.

Thm (Chain Rule)

① If $z=f(x,y)$ where $x=x(t)$ and $y=y(t)$, $\boxed{\frac{dz}{dt}=\frac{\partial z}{\partial x}\frac{dx}{dt}+\frac{\partial z}{\partial y}\frac{dy}{dt}}$

② If $z=f(x,y)$ where $x=x(s,t)$ and $y=y(s,t)$,

$$\boxed{\frac{\partial z}{\partial s}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial s}} \quad \text{and} \quad \boxed{\frac{\partial z}{\partial t}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial t}}$$

Ex.4 Find $\frac{dz}{dt}$ when $z=\ln(x)+\cos(y)+xy^2$, $x=1+\sqrt{t}$, and $y=e^t$.

Note: we could substitute to get $z(t)=\ln(1+\sqrt{t})+\cos(e^t)+(1+\sqrt{t})e^{2t}$.
However, finding $\frac{\partial z}{\partial t}$ would be very messy.

To use the new chain rule, we first compute all the pieces:

$$\frac{\partial z}{\partial x}=\frac{1}{x}+y^2 \qquad \frac{\partial z}{\partial y}=-\sin(y)+2xy$$

$$\frac{dx}{dt}=\frac{1}{2\sqrt{t}} \qquad \frac{dy}{dt}=e^t$$

Then $\frac{dz}{dt}=\frac{\partial z}{\partial x}\frac{dx}{dt}+\frac{\partial z}{\partial y}\frac{dy}{dt}=\boxed{\left(\frac{1}{x}+y^2\right)\left(\frac{1}{2\sqrt{t}}\right)+(2xy-\sin(y))e^t}$.

Now, we could substitute to have $\frac{dz}{dt}$ in terms of only t.

$$\frac{dz}{dt}=\left(\frac{1}{1+\sqrt{t}}+e^{2t}\right)\left(\frac{1}{2\sqrt{t}}\right)+(2(1+\sqrt{t})e^t-\sin(e^t))e^t$$

This should match what we would get if we substituted to write $z(t)$ <u>before</u> taking the derivative.

Ex.5 Find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$ for $z=e^{xy}$, $x=t\ln(s)$, $y=t^2+st$

Again, we compute the pieces:

$$\frac{\partial z}{\partial x}=ye^{xy} \qquad \frac{\partial z}{\partial y}=xe^{xy}$$

$$\frac{\partial x}{\partial s}=\frac{t}{s} \qquad \frac{\partial y}{\partial s}=t$$

$$\frac{\partial x}{\partial t}=\ln(s) \qquad \frac{\partial y}{\partial t}=2t+s$$

$$\frac{\partial z}{\partial s}=(ye^{xy})\left(\frac{t}{s}\right)+xe^{xy}(t)=\boxed{\left(\frac{y}{s}+x\right)te^{xy}}$$

$$\frac{\partial z}{\partial t}=(ye^{xy})(\ln(s))+xe^{xy}(2t+s)=\boxed{(y\ln(s)+x(2t+s))e^{xy}}$$

Thm (Implicit Differentiation)

If $F(x,y)=0$ and $F_y \neq 0$, then $\boxed{\frac{dy}{dx} = -\frac{F_x}{F_y}}$

Why?

$$F(x,y)=0$$
$$\frac{\partial}{\partial x}[F(x,y)]=0$$
$$\underbrace{\cancel{\left(\frac{dx}{dx}\right)}}_{=1} F_x + \left(\frac{dy}{dx}\right)F_y = 0 \quad \text{by chain rule}$$
$$\left(\frac{dy}{dx}\right)F_y = -F_x$$
$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

Note: We are assuming y is a differentiable function of x. This is useful when working with $x^2+y^2=1$ and similar curves because $y=\pm\sqrt{1-x^2}$ can't be solved explicitly for a single function everywhere.

Ex. 6 Find $\frac{dy}{dx}$ for $x^2y^2 = \sin(x) + e^y$.

$F(x,y) = x^2y^2 - \sin(x) - e^y$ (or $F(x,y) = \sin(x) + e^y - x^2y^2$)

$F_x(x,y) = 2xy^2 - \cos(x)$

$F_y(x,y) = 2x^2y - e^y$

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{2xy^2-\cos(x)}{2x^2y-e^y}$$
$$= \boxed{\frac{\cos(x)-2xy^2}{2x^2y-e^y}}$$

Ex. 7 Use implicit differentiation to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for
$e^{z^2} = yz + z\ln x$

Note: We can write $F(x,y,z) = 0$

$$\frac{\partial}{\partial x}[F(x,y,z)] = 0$$
$$\underbrace{\cancel{\frac{\partial x}{\partial x}}}_{=1} F_x + \underbrace{\cancel{\frac{\partial y}{\partial x}}}_{=0} F_y + \frac{\partial z}{\partial x}F_z = 0$$
$$\Rightarrow \frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$$

Similarly, $\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$.

We can assume z is a function of x and y and that x and y are independent of each other. If we could write $y = y(x)$, then this should be $\frac{dz}{dx}$. Instead, we know $\frac{\partial y}{\partial x} = 0$.

$$F(x,y,z) = e^{z^2} - yz - z\ln(x)$$

$$F_z = 2ze^{z^2} - y - \ln(x)$$

$$F_y = -z$$

$$F_x = -\frac{z}{x}$$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{-z/x}{2ze^{z^2} - y - \ln(x)} = \boxed{\frac{z}{x(2ze^{z^2} - y - \ln(x))}}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{-z}{2ze^{z^2} - y - \ln(x)} = \boxed{\frac{z}{2ze^{z^2} - y - \ln(x)}}$$

Ex. 8 Show that $u(x,t) = f(x+at) + g(x-at)$ solves $u_{tt} = a^2 u_{xx}$.

$$u_t = af'(x+at) - ag'(x-at)$$

$$u_{tt} = a^2 f''(x+at) + a^2 g''(x-at) = a^2\left(\underbrace{f''(x+at) + g''(x-at)}_{=u_{xx}}\right)$$

$$u_x = f'(x+at) + g'(x-at)$$

$$u_{xx} = f''(x+at) + g''(x-at)$$

$u_{tt} = a^2 u_{xx}$ ✓

Ex. 9 The volume of a pyramid with a square base x units on a side and a height of h is $V = \frac{1}{3}x^2 h$.

(a) Assume x and t are functions of t. Find $V'(t)$.

$$V'(t) = \frac{dV}{dt} = \frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt} \qquad \frac{\partial V}{\partial x} = \frac{2}{3}xh \qquad \frac{\partial V}{\partial h} = \frac{1}{3}x^2$$

$$= \left(\frac{2}{3}xh\right)\frac{dx}{dt} + \left(\frac{1}{3}x^2\right)\frac{dh}{dt}$$

$$\boxed{V'(t) = \frac{2}{3}x(t)h(t)x'(t) + \frac{1}{3}(x(t))^2 h'(t)}$$

(b) Suppose $x = \frac{t}{t+1}$ and $h = \frac{1}{t+1}$ for $t \geq 0$. Use (a) to find $V'(t)$.

$$x'(t) = \frac{(t+1) - t}{(t+1)^2} = \frac{1}{(t+1)^2}$$

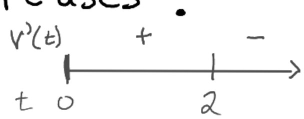
$$h'(t) = \frac{-1}{(t+1)^2}$$

$$V'(t) = \frac{2}{3}\left(\frac{t}{t+1}\right)\left(\frac{1}{t+1}\right)\left(\frac{1}{(t+1)^2}\right) + \frac{1}{3}\left(\frac{t}{t+1}\right)^2\left(-\frac{1}{(t+1)^2}\right)$$

$$= \frac{1}{3(t+1)^4}(2t - t^2) = \boxed{\frac{2t - t^2}{3(t+1)^4}}$$

(c) Does the volume of the pyramid in part (b) increase or decrease as t increases?

$$0 = 2t - t^2 = t(2-t)$$



$$\boxed{\begin{array}{l}\text{Increases for } 0 \leq t \leq 2 \\ \text{Decreases for } t \geq 2\end{array}}$$