§ 15.5 Directional Derivatives and the Gradient

$f_x(x,y)$ is the rate of change of f with respect to x, i.e. how much does a change in x affect the value of f at (x,y) if y is held constant. Similarly, $f_y(x,y)$ is how much a change in y affects the value of f if x is held constant.
What happens if we change both x and y?
It depends in which direction we change them.
The maximum contribution from f_x is in the direction $\langle 1,0\rangle$ and the maximum contribution from f_y is in the direction $\langle 0,1\rangle$. Therefore, the maximum change in $f(x,y)$ is in the direction $f_x\langle 1,0\rangle + f_y\langle 0,1\rangle = \langle f_x, f_y\rangle$.

Def. The gradient of $f(x,y)$ is $\boxed{\nabla f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle}$.
$|\nabla f(x,y)|$ is the maximum rate of change (or steepest ascent) of f. $\nabla f(x,y)$ is the direction in which this change occurs.

We can choose to vary x and y in any direction, say $\vec{u} = \langle u_1, u_2\rangle$. Then we use u_1 to weight the change in f with respect to x and u_2 to weight the change in f with respect to y. Then $f_x u_1 + f_y u_2$ would be in the change in f in the direction $\vec{u}$. However, if we consider two vectors in the same direction with different magnitudes (like $\langle 2,1\rangle$ and $\langle 4,2\rangle$), we get different magnitudes for the change in f in this direction. For this reason, we will use unit vectors in the direction we examine.

Def. The directional derivative of f in the direction of the unit vector $\vec{u}$ is $\boxed{D_{\vec{u}} f(x,y) = \nabla f(x,y) \bullet \vec{u}} = u_1 f_x(x,y) + u_2 f_y(x,y)$.
$D_{\vec{u}} f$ is the rate of change of f in the direction $\vec{u}$ at any point (x,y).

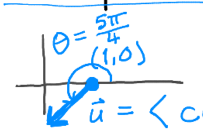
Note: $D_{\vec{u}} f(x,y) = \nabla f(x,y) \bullet \vec{u}$
$= |\nabla f(x,y)|\,|\vec{u}| \cos\theta$

Steepest ascent (maximum increase of f) is when $\cos\theta = 1$, i.e. when ∇f and $\vec{u}$ are in the same direction.
Steepest descent (maximum decrease of f) is when $\cos\theta = -1$, i.e. when ∇f and $\vec{u}$ are in opposite directions.
There is no change in f when $\cos\theta = 0$, i.e. ∇f and $\vec{u}$ orthogonal.

Ex.1 Find $\nabla f(x,y)$ when $f(x,y) = x^2y + e^x + \cos(y)$.

$$\nabla f(x,y) = \left\langle \frac{\partial}{\partial x}\left[x^2y + e^x + \cos(y)\right], \frac{\partial}{\partial y}\left[x^2y + e^x + \cos(y)\right] \right\rangle$$
$$= \boxed{\langle 2xy + e^x, \; x^2 - \sin(y) \rangle}$$

Ex.2 Find the directional derivative of $f(x,y) = x^2y + e^{xy} + x^3$ at the point $(1,0)$ in the direction $\theta = \frac{5\pi}{4}$.



$\vec{u} = \langle \cos\theta, \sin\theta \rangle$
$= \left\langle -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right\rangle$

$f_x = 2xy + ye^{xy} + 3x^2$
$f_y = x^2 + xe^{xy}$

$$D_{\vec{u}} f(x,y) = \langle 2xy + ye^{xy} + 3x^2, \; x^2 + xe^{xy} \rangle \cdot \left\langle -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right\rangle$$
$$D_{\vec{u}} f(1,0) = \langle 2(1)(0) + (0)e^{(1)(0)} + 3(1)^2, \; 1^2 + (1)e^{(1)(0)} \rangle \cdot \left\langle -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right\rangle$$
$$= \langle 3, 2 \rangle \cdot \left\langle -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right\rangle$$
$$= -\frac{3\sqrt{2}}{2} - \frac{2\sqrt{2}}{2} = \boxed{-\frac{5\sqrt{2}}{2}}$$

Ex.3 Find all points at which the direction of steepest ascent of $g(s,t) = t^2 - s^2 + st + s$ is $3\vec{i} - \vec{j}$.

First, find $\nabla g = \langle g_s, g_t \rangle = \langle -2s + t + 1, \; 2t + s \rangle$.
Now, solve $\nabla g = k\langle 3, -1 \rangle$

since we can have any magnitude

$$\begin{cases} -2s + t + 1 = 3k \\ 2t + s = -k \end{cases} \longrightarrow \quad s = -k - 2t$$

$$-2(-k - 2t) + t + 1 = 3k$$
$$2k + 4t + t + 1 = 3k$$
$$5t = k - 1$$
$$t = \frac{k-1}{5}$$

$$s = -k - 2t$$
$$s = -k - 2\left(\frac{k-1}{5}\right)$$
$$s = \frac{-5k - 2k + 2}{5}$$
$$s = \frac{-7k + 2}{5}$$

$$\boxed{\text{All points } \left(\frac{2-7k}{5}, \frac{k-1}{5}\right)}$$

Ex.4 Find the unit vectors in the direction of steepest descent and steepest ascent of $h(x,y,z) = x^2y - ye^x + z^2 + 2e^{yz}$ at $(0,1,0)$. Find a vector that points in the direction of no change at this point.

$$\nabla h = \langle 2xy - ye^x,\ x^2 - e^x + 2ze^{yz},\ 2z + 2ye^{yz} \rangle$$
$$\nabla h(0,1,0) = \langle 0 - 1\cdot e^0,\ 0 - e^0 + 2(0)e^0,\ 2(0) + 2(1)e^0 \rangle = \langle -1, -1, 2 \rangle$$

Since $D_{\vec{u}} \nabla h = \vec{u} \cdot \nabla h = |\vec{u}||\nabla h| \cos\theta$, steepest ascent is when $\cos\theta = 1$ and steepest descent is when $\cos\theta = -1$. In other words, steepest descent is in the opposite direction of ∇h.

Steepest ascent: $\frac{\nabla h}{|\nabla h|} = \frac{\langle -1,-1,2 \rangle}{\sqrt{6}} = \boxed{\left\langle -\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}} \right\rangle}$

Steepest descent: $-\frac{\nabla h}{|\nabla h|} = \boxed{\left\langle \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}} \right\rangle}$

No change $\Rightarrow$ need $\cos\theta = 0$, so need $\langle a,b,c \rangle \cdot \nabla h = 0$

$$\Rightarrow -\frac{a}{\sqrt{6}} - \frac{b}{\sqrt{6}} + \frac{2c}{\sqrt{6}} = 0$$

If $a = b = c$, $-\frac{1}{\sqrt{6}} - \frac{1}{\sqrt{6}} + \frac{2}{\sqrt{6}} = 0$.

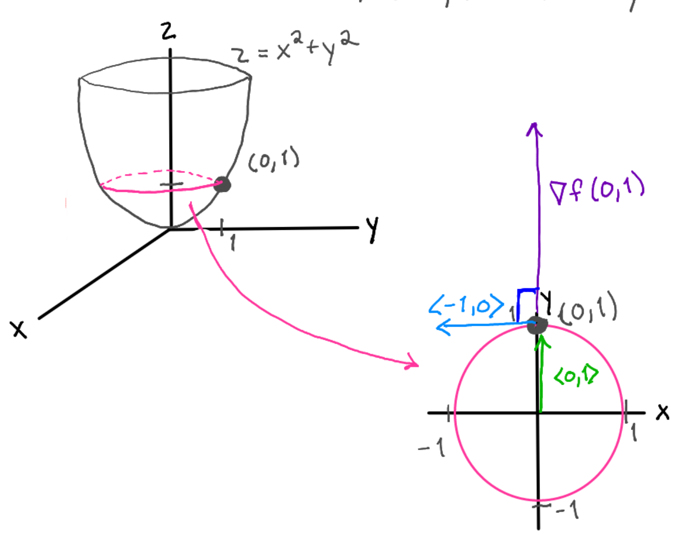
A vector pointing in the direction of no change is $\boxed{\langle 1,1,1 \rangle}$.

Thm Gradient and Level Curves

The line tangent to the level curve of f at (a,b) is orthogonal to the gradient $\nabla f(a,b)$ if $\nabla f(a,b) \neq \vec{0}$.

What does this mean?

Consider $z = f(x,y) = x^2 + y^2$ at $(0,1)$. $f(0,1) = 1$, so we look at the level curve $x^2 + y^2 = 1$. We can parameterize this with $\vec{r}(t) = \langle \cos t, \sin t \rangle$ where $\vec{r}(\frac{\pi}{2}) = \langle 0, 1 \rangle$. Then $\vec{r}\,'(t) = \langle -\sin t, \cos t \rangle$ and $\vec{r}\,'(\frac{\pi}{2}) = \langle -1, 0 \rangle$.



Now, $\nabla f = \langle 2x, 2y \rangle$, so $\nabla f(0,1) = \langle 0, 2 \rangle$.

$\nabla f(0,1) \cdot \vec{r}\,'(\frac{\pi}{2}) = 0$

$\Rightarrow$ gradient is orthogonal to tangent at $(0,1)$.

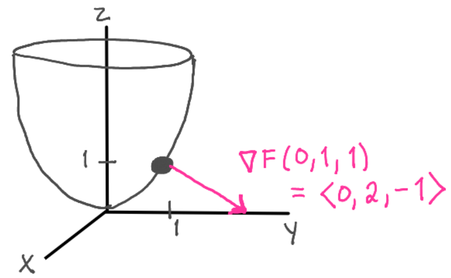
§15.6 Tangent Planes and Linear Approximations

Def. The tangent plane to the surface $F(x,y,z)=0$ at $P_0(a,b,c)$ is the plane passing through P_0 and orthogonal to $\nabla F(a,b,c)$.

The plane is given by $\boxed{F_x(a,b,c)(x-a)+F_y(a,b,c)(y-b)+F_z(a,b,c)(z-c)=0}$

OR $\nabla F(a,b,c) \bullet \langle x-a,\ y-b,\ z-c\rangle = 0$

Let's look at $z=x^2+y^2$ at $(0,1)$ again. For the tangent plane, we write this as $F(x,y,z)=x^2+y^2-z=0$ at $(0,1,1)$.
Then $\nabla F = \langle 2x, 2y, -1\rangle$ and $\nabla F(0,1,1)=\langle 0,2,-1\rangle$.



When we discussed the gradient for $z=f(x,y)$, we draw the gradient in the xy-plane. The gradient ∇f is perpendicular to the tangent line to the level curve at (a,b)

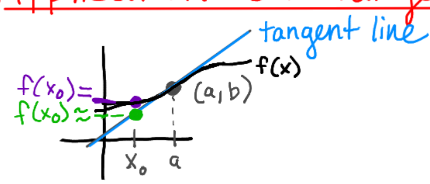
Now, we are dealing with $F(x,y,z)=0$. The gradient ∇F is in 3D space and is perpendicular to the tangent plane to the surface at (a,b,c).
This is why the equation of the tangent plane is $\nabla F(a,b,c) \bullet \overrightarrow{P_0P} = 0$ where $P_0(a,b,c)$ is the point on the surface and $P(x,y,z)$ is any point on the tangent plane.

Ex.5 Find an equation of the tangent plane to $z^2=3e^x+xy^2+y$ at the point $(0,1,-2)$.

$$F(x,y,z)=3e^x+xy^2+y-z^2=0$$
$$\nabla F(x,y,z)=\langle 3e^x+y^2,\ 2xy+1,\ -2z\rangle$$
$$\nabla F(0,1,-2)=\langle 3+1,\ 2(0)(1)+1,\ -2(-2)\rangle=\langle 4,1,4\rangle$$

Plane: $4(x-0)+1(y-1)+4(z+2)=0$
$$\boxed{4x+(y-1)+4(z+2)=0}$$

Application of Tangent Planes



In 2D, we can estimate values of $f(x)$ near $x=a$ by using the tangent line.

In 3D, we will use a tangent plane to estimate values of $f(x,y)$ near (a,b).

Def. The <u>linear approximation</u> to the surface $z=f(x,y)$ at the point $(a,b,f(a,b))$ is given by $\boxed{L(x,y)=f_x(a,b)\cdot(x-a)+f_y(a,b)\cdot(y-b)+f(a,b)}$

For a function of 3 variables, $w=f(x,y,z)$ at the point $(a,b,c,f(a,b,c))$, we have

$$\boxed{L(x,y,z)=f_x(a,b,c)(x-a)+f_y(a,b,c)(y-b)+f_z(a,b,c)(z-c)+f(a,b,c)}$$

Ex.6 Find the linear approximation of $f(x,y,z)=(x^2+y^2+z^2)^3$ at $(1,2,2)$ and use it to approximate $\left(\left(\frac{3}{4}\right)^2+\left(\frac{7}{3}\right)^2+\left(\frac{15}{8}\right)^2\right)^3$.

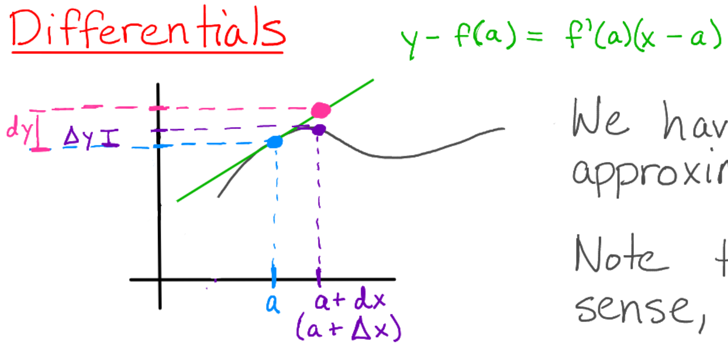
$$\nabla f(x,y,z)=\langle 6x(x^2+y^2+z^2)^2,\ 6y(x^2+y^2+z^2)^2,\ 6z(x^2+y^2+z^2)^2\rangle$$
$$\nabla f(1,2,2)=\langle 6(9)^2,\ 12(9)^2,\ 12(9)^2\rangle=\langle 486,972,972\rangle$$

$$f(1,2,2)=(9)^3=729$$

$$\boxed{L(x,y,z)=486(x-1)+972(y-2)+972(z-2)+729}$$

$$L\left(\tfrac{3}{4},\tfrac{7}{3},\tfrac{15}{8}\right)=486\left(-\tfrac{1}{4}\right)+972\left(\tfrac{1}{3}\right)+972\left(-\tfrac{1}{8}\right)+729$$
$$=\boxed{810}$$

Differentials



We have $\Delta y=f(a+dx)-f(a)$ and approximate $\Delta y\approx dy=f'(a)\cdot dx$

Note that $f'(x)=\frac{dy}{dx}$, so in some sense, this is a rearrangement.

Def. Let $z=f(x,y)$. The <u>exact change in z, Δz,</u> from (a,b) to $(a+dx,b+dy)$ is $\boxed{\Delta z=f(a+dx,b+dy)-f(a,b)}$

The <u>differential, dz,</u> to approximate Δz is

$$\boxed{dz\approx f_x(a,b)\,dx+f_y(a,b)\,dy}$$

Ex.7 If $z = x^2 + y^2$ and (x,y) changes from $(-1,1)$ to $(-\frac{2}{3}, \frac{4}{3})$, compare the values of Δz and dz.

$$\begin{aligned}\Delta z &= f(-\tfrac{2}{3}, \tfrac{4}{3}) - f(-1,1) \\ &= \left((-\tfrac{2}{3})^2 + (\tfrac{4}{3})^2\right) - \left((-1)^2 + (1)^2\right) \\ &= \left(\tfrac{4}{9} + \tfrac{16}{9}\right) - (1+1) \\ &= \tfrac{20}{9} - 2 \\ &= \boxed{\tfrac{2}{9}}\end{aligned}$$

$$\nabla f(x,y) = \langle 2x, 2y \rangle$$
$$\nabla f(-1,1) = \langle -2, 2 \rangle$$

$$\begin{aligned}dz &= -2\left(-\tfrac{2}{3} - (-1)\right) + 2\left(\tfrac{4}{3} - 1\right) \\ &= -2\left(\tfrac{1}{3}\right) + 2\left(\tfrac{1}{3}\right) \\ &= -\tfrac{2}{3} + \tfrac{2}{3} \\ &= \boxed{0}\end{aligned}$$