§15.7 Maximum / Minimum Problems

Def. $f(a,b)$ is a local maximum of f if there is a small region R around (a,b) where $f(a,b) \geq f(x,y)$ for all (x,y) in R.
$f(a,b)$ is a local minimum of f if there is a small region R around (a,b) where $f(a,b) \leq f(x,y)$ for all (x,y) in R.

(a,b) is a critical point of f if either

① $f_x(a,b) = f_y(a,b) = 0$ or

② at least one of f_x and f_y does not exist.

f has a saddle point at (a,b) if $f(a,b)$ is neither a local min nor a local max. This means there are points (x,y) near (a,b) for which $f(x,y) > f(a,b)$ and points for which $f(x,y) < f(a,b)$.

Once we find the critical points of a function, we can classify them as local mins, local maxes, or saddle points using the Second Derivative Test (2^{nd} DT)

Thm Second Derivative Test

Let $D(x,y) = f_{xx}(x,y)\, f_{yy}(x,y) - [f_{xy}(x,y)]^2$.
We can classify a critical point (a,b) of f (where $f_x(a,b) = f_y(a,b) = 0$) as follows:

1) If $D(a,b) > 0$ and $f_{xx}(a,b) < 0$, then $f(a,b)$ is a local max.
2) If $D(a,b) > 0$ and $f_{xx}(a,b) > 0$, then $f(a,b)$ is a local min.
3) If $D(a,b) < 0$, then f has a saddle point at (a,b).
4) If $D(a,b) = 0$, then the test is inconclusive at (a,b).

Remark 1 $D(x,y)$ is called the discriminant and is the determinant of the matrix of second partials $\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$ if $f_{xy} = f_{yx}$.

Remark 2 If the 2^{nd} DT is inconclusive, f could have a local min, local max, or saddle point (a,b). We need to use other tools like graphing to classify (a,b).

Ex.1 Find the critical points of $f(x,y) = x^3 + 6xy - 6x + y^2 - 2y$.

We need to find (a,b) so that $f_x(a,b) = f_y(a,b) = 0$ or one is undefined.

$$\begin{cases} f_x = 3x^2 + 6y - 6 = 0 & ① \\ f_y = 6x + 2y - 2 = 0 & ② \end{cases}$$

(Both defined everywhere.)

$$② \Rightarrow 2y = 2 - 6x$$
$$y = 1 - 3x \longrightarrow \text{Plug in to } ① \rightarrow 3x^2 + 6(1-3x) - 6 = 0$$
$$3x^2 - 18x = 0$$
$$\underbrace{3x}_{x=0}\underbrace{(x-6)}_{x=6} = 0$$

Now that we have x-values, plug them into $y = 1-3x$ to find the corresponding y-values.

	$x=0$	$x=6$
$y=1-3x$	$y=1$	$y=-17$

$$\boxed{\text{Critical points: } (0,1) \text{ and } (6,-17)}$$

Ex.2 Find the critical points of $f(x,y) = e^{8x^2y^2 - 24x^2 - 8xy^4}$.

$$f_x = (16xy^2 - 48x - 8y^4)\underbrace{e^{8x^2y^2-24x^2-8xy^4}}_{\text{always} > 0} = 0$$

$$f_y = (16x^2y - 32xy^3)\underbrace{e^{8x^2y^2-24x^2-8xy^4}}_{\text{always} > 0} = 0$$

$$\Rightarrow \text{Solve } \begin{cases} 16xy^2 - 48x - 8y^4 = 0 & ① \\ 16x^2y - 32xy^3 = 0 & ② \end{cases}$$

$$② \Rightarrow \underbrace{16xy}\underbrace{(x-2y^2)} = 0$$

$\underline{x=0}$

① $0 - 0 - 8y^4 = 0$

$\Rightarrow y = 0$

$\underline{y=0}$

$0 - 48x - 0 = 0$

$\Rightarrow x = 0$

$\underline{x = 2y^2}$

$0 = 16(2y^2)y^2 - 48(2y^2) - 8y^4$

$0 = 32y^4 - 96y^2 - 8y^4$

$0 = \underbrace{24y^2}_{y=0}\underbrace{(y^2-4)}_{y=\pm 2}$

$y = \pm 2 \Rightarrow x = 2y^2 = 8$

$$\boxed{\text{Critical points: } (0,0), (8,-2), (8,2)}$$

Ex.3 Find and classify the critical points of $f(x,y)=x^2+xy^2-2x+1$

$$\begin{cases} f_x = 2x+y^2-2=0 \\ f_y = 2xy=0 \end{cases} \Rightarrow \text{Critical points are } (0,\sqrt{2}),\ (0,-\sqrt{2}),\ (1,0)$$

$f_{xx}=2$
$f_{yy}=2x$
$f_{xy}=2y$

$$D(x,y)=f_{xx}f_{yy}-f_{xy}^2 = 2(2x)-(2y)^2 = 4x-2y^2$$

(a,b)	$D(a,b)$	$f_{xx}(a,b)$	Classification
$(0,\sqrt{2})$	$0-8=-8$	—	Saddle Point
$(0,-\sqrt{2})$	$0-8=-8$	—	Saddle Point
$(1,0)$	$4-0=4$	$=2$	Local Min

Note: At the saddle point $(0,\sqrt{2})$,
$\nabla f(0,\sqrt{2}) = \langle 2(0)+(\sqrt{2})^2-2,\ 2(0)\sqrt{2}\rangle = \langle 0,0\rangle$
because there may be multiple directions of steepest ascent.

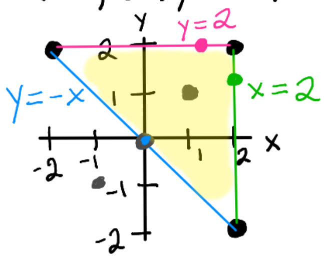
* Absolute Min and Max

Def. On a bounded region R, $f(a,b)$ is an absolute maximum if $f(a,b)\geq f(x,y)$ for all (x,y) in R.
$f(a,b)$ is an absolute minimum if $f(a,b)\leq f(x,y)$ for all (x,y) in R.

To find absolute min/max on a region R:
1. Find the critical points of f that are in R.
2. Find the minimum and maximum values on the boundary of R. The boundary or constraint can be used to write $f(x,y)$ in terms of 1 variable. Then take the derivative, set equal to 0 and solve to see where on the boundary the min or max occurs.
3. Evaluate f at the points (a,b) found in 1 and 2.

The largest value of f is the max, and the smallest is the min.
Do NOT use the 2nd DT as the points on the boundary may not be critical points.

Ex.4 Find the absolute min and max of $f(x,y) = x^4 + y^4 - 4xy$ on the closed triangular region with vertices $(-2,2)$, $(2,2)$, and $(2,-2)$.



① Critical Points: $\begin{cases} f_x = 4x^3 - 4y = 0 \\ f_y = 4y^3 - 4x = 0 \end{cases}$

gives critical $(0,0)$, $(1,1)$, and ~~$(-1,-1)$~~
(on $y=-x$) (inside) ~~outside~~

② $\underline{y=2}$: $f(x,2) = x^4 + 16 - 8x$
$f'(x) = 4x^3 - 8 = 0$
$x = \sqrt[3]{2} \approx 1.26$

$\underline{x=2}$: $f(2,y) = 2^4 + y^4 - 8y$
$f'(y) = 4y^3 - 8 = 0$
$y = \sqrt[3]{2} \approx 1.26$

$\underline{y=-x}$: $f(x,-x) = x^4 + x^4 + 4x^2$
$f'(x) = 8x^3 + 8x = 0$
$\underbrace{8x}_{x=0}\underbrace{(x^2+1)}_{\neq 0} = 0$

③ Evaluate f:

(a,b)	$f(x,y) = x^4 + y^4 - 4xy$
$(0,0)$	0
$(1,1)$	$1+1-4 = -2$
$(\sqrt[3]{2}, 2)$	$2\sqrt[3]{2} + 16 - 8\sqrt[3]{2} \approx 8.44$
$(2, \sqrt[3]{2})$	$16 + 2\sqrt[3]{2} - 8\sqrt[3]{2} \approx 8.44$

$\boxed{\begin{array}{l} \text{Abs. Max of } 16 - 6\sqrt[3]{2} \\ \text{Abs. Min of } -2 \end{array}}$

Ex.5 Find the points on the cone $z^2 = x^2 + y^2$ closest to the point $(4,2,0)$.

The minimum distance $D = \sqrt{(x-4)^2 + (y-2)^2 + z^2}$ is attained when $d = D^2 = (x-4)^2 + (y-2)^2 + z^2$ is minimized.
We can treat $z^2 = x^2 + y^2$ as a <u>constraint</u> rather than a boundary, so we plug this into d and find the critical points of f subject to this relationship between x, y, and z. Without this

restriction on values of x and y (i.e. $x^2+y^2=z^2$), we would get different critical points for the local extrema. Meaning, we still do not use the 2^{nd} DT to classify. $d = (x-4)^2+(y-2)^2+x^2+y^2$

① Critical Points: Solve $\begin{cases} d_x = 2(x-4)+2x = 0 \\ d_y = 2(y-2)+2y \end{cases}$ to get $(2,1)$

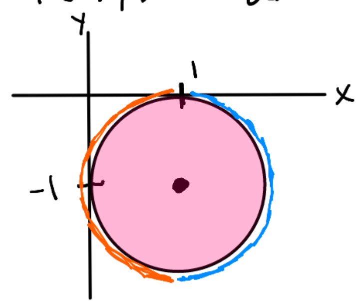
② $z^2 = x^2+y^2$

$z^2 = 4+1$

$z = \pm\sqrt{5}$

③ $\boxed{\text{Minimum distance at points } (2,1,\pm\sqrt{5})}$

<u>Ex.6</u> Find the absolute minimum and maximum of $f(x,y) = -2x^2+4x-3y^2-6y-1$ on $R=\{(x,y):(x-1)^2+(y+1)^2\le 1\}$.



① Critical points: Solve $\begin{cases} -4x+4=0 \\ -6y-6=0 \end{cases}$ to get $(1,-1)$.

② On the boundary, if we work in terms of x and y, we have to treat it as two functions: $x = 1-\sqrt{1-(y+1)^2}$

$x = 1+\sqrt{1-(y+1)^2}$

$f(1-\sqrt{1-(y+1)^2}, y) = -2(1-\sqrt{1-(y+1)^2})^2+4(1-\sqrt{1-(y+1)^2})-3y^2-6y-1$

$f(1+\sqrt{1-(y+1)^2}, y) = -2(1+\sqrt{1-(y+1)^2})^2+4(1+\sqrt{1-(y+1)^2})-3y^2-6y-1$

However, these derivatives are annoying to take. Instead, we can parameterize the circle instead:

We want to change $\underbrace{(x-1)^2}+\underbrace{(y+1)^2}=1$ to $\underbrace{\cos^2 t}+\underbrace{\sin^2 t}=1$

$x-1 = \cos t$ $\quad$ $y+1 = \sin t$

$x = \cos t+1$ $\quad$ $y = \sin t - 1$ $\quad$ $0 \le t \le 2\pi$

Then $f(t) = f(x(t),y(t)) = f(\cos t+1, \sin t-1)$

$= -2(\cos t+1)^2 + 4(\cos t+4) - 3(\sin t-1)^2 - 6(\sin t-1) - 1$

$\Rightarrow f'(t) = -4(\cos t \cancel{+1})(-\sin t) - \cancel{4\sin t} - 6(\sin t \cancel{-1})\cos t - \cancel{6\cos t}$

$0 = -10\sin t\cos t$

$0 = -10\left(\frac{\sin(2t)}{2}\right)$

$0 = \sin(2t)$

$$2t = 0 + \pi n$$
$$t = \frac{\pi}{2} \cdot n$$

$$\Rightarrow t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi, \ldots$$

$t = 0$	$t = \frac{\pi}{2}$	$t = \pi$	$t = \frac{3\pi}{2}$	$t = 2\pi$
$x = 2$	$x = 1$	$x = 0$	$x = 1$	$x = 2$
$y = -1$	$y = 0$	$y = -1$	$y = -2$	$y = -1$

3

(x,y)	$f(x,y) = -2x^2 + 4x - 3y^2 - 6y - 1$
$(2,-1)$	$-8 + 8 - 3 + 6 - 1 = 2$
$(1,0)$	$-2 + 4 - 1 = 1$
$(0,-1)$	$-3 + 6 - 1 = 2$
$(1,-2)$	$-2 + 4 - 12 + 12 - 1 = \boxed{1 = \text{min}}$
$(1,-1)$	$-2 + 4 - 3 + 6 - 1 = \boxed{4 = \text{max}}$