MA261 Exam 1 Review

Chapter 12: Vectors and the Geometry of Space

§**12.1** Three-Dimensional Coordinate Systems

Points in three-dimensional space are represented by (a, b, c) where $x = a, y = b, z = c$. Three-dimensional space can be divided into eight octants based on each of the dimensions x, y, and z being positive or negative. The **first octant** is where all three variables are positive.

The distance between points $P(x_1, y_1, z_1)$ and $P(x_2, y_2, z_2)$ is

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

The equation of a sphere with center $C(h, k, l)$ and radius r is

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$$

We can see from the equation of the sphere that every point (x, y, z) is a distance of r from the center of the sphere.
The **coordinate planes** are the xy-plane (when $z = 0$), the yz-plane (when $x = 0$), and the xz-plane (when $y = 0$).

§**12.2** Vectors

A **vector** is a quantity with magnitude and direction. We usually encounter two-dimensional and three-dimensional vectors, but vectors can have any number of dimensions.

The **displacement vector** from point $A(x_1, y_1, z_1)$ to point $B(x_2, y_2, z_2)$ is

$$\overrightarrow{AB} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

Notice that $\overrightarrow{AB} = -\overrightarrow{BA}$, so the vectors lie in the same place in space but point in opposite directions.

The **length** or **magnitude** of a vector $\mathbf{v} = \langle a, b, c \rangle$ is

$$|\mathbf{v}| = \sqrt{a^2 + b^2 + c^2}$$

If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ and k is a scalar,

$$\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$$

$$k\mathbf{a} = \langle ka_1, ka_2, ka_3 \rangle$$

Properties of Vectors

If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors, and c and d are scalars, then

1. $\mathbf{a}+\mathbf{b}=\mathbf{b}+\mathbf{a}$
2. $\mathbf{a}+(\mathbf{b}+\mathbf{c})=(\mathbf{a}+\mathbf{b})+\mathbf{c}$
3. $\mathbf{a}+\mathbf{0}=\mathbf{a}$
4. $\mathbf{a}+(-\mathbf{a})=\mathbf{0}$
5. $c(\mathbf{a}+\mathbf{b})=c\mathbf{a}+c\mathbf{b}$
6. $(c+d)\mathbf{a}=c\mathbf{a}+d\mathbf{a}$
7. $(cd)\mathbf{a}=c(d\mathbf{a})$
8. $1\mathbf{a}=\mathbf{a}$

We often write vectors in terms of the **standard basis vectors** $\hat{\mathbf{i}}, \hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ where $\hat{\mathbf{i}}=\langle 1,0,0\rangle$, $\hat{\mathbf{j}}=\langle 0,1,0\rangle$, and $\hat{\mathbf{k}}=\langle 0,0,1\rangle$. Note that all of these are **unit vectors** meaning that the magnitude of each is 1. We can write any vector $\mathbf{v}=\langle a,b,c\rangle$ as $\mathbf{v}=a\hat{\mathbf{i}}+b\hat{\mathbf{j}}+c\hat{\mathbf{k}}$. We can find the unit vector $\mathbf{u}$ in the direction of vector $\mathbf{v}$ with

$$\mathbf{u}=\frac{\mathbf{v}}{|\mathbf{v}|}$$

§**12.3** The Dot Product

The **dot product** between vectors $\mathbf{a}=\langle a_1,a_2,a_3\rangle$ and $\mathbf{b}=\langle b_1,b_2,b_3\rangle$ is

$$\mathbf{a}\cdot\mathbf{b}=a_1b_1+a_2b_2+a_3b_3$$

Notice that the dot product is a scalar.

Properties of the Dot Product

If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors, and c is a scalar, then

1. $\mathbf{a}\cdot\mathbf{a}=|\mathbf{a}|^2$
2. $\mathbf{a}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{a}$
3. $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c})=\mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c}$
4. $(c\mathbf{a})\cdot\mathbf{b}=c(\mathbf{a}\cdot\mathbf{b})=\mathbf{a}\cdot(c\mathbf{b})$
5. $\mathbf{0}\cdot\mathbf{a}=0$

If θ is the angle between $\mathbf{a}$ and $\mathbf{b}$,

$$\mathbf{a}\cdot\mathbf{b}=|\mathbf{a}||\mathbf{b}|\cos(\theta)$$

Notice that when $\theta = \frac{\pi}{2} + \pi n$ where n is any integer, $\cos(\theta) = 0$, so the dot product is 0. This means that two vectors are orthogonal (or perpendicular) if and only if their dot product is 0.

§**12.4** The Cross Product

If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, the **cross product** gives a vector that is perpendicular to both **a** and **b** and is found by

$$\mathbf{a} \times \mathbf{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle$$

We can get this formula using determinants as well. Notice the minus sign in front of the middle term.

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\ &= \hat{\mathbf{i}} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \hat{\mathbf{j}} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \hat{\mathbf{k}} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \\ &= (a_2b_3 - a_3b_2)\hat{\mathbf{i}} - (a_1b_3 - a_3b_1)\hat{\mathbf{j}} + (a_1b_2 - a_2b_1)\hat{\mathbf{k}} \\ &= \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle \end{aligned}$$

If θ is the angle between **a** and **b**,

$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}| \sin(\theta)$$

Notice that when $\theta = \pi n$ where n is any integer, $\sin(\theta) = 0$, so the cross product is 0. This means that two vectors are parallel if and only if their cross product is 0.

Properties of the Cross Product

If **a**, **b**, and **c** are vectors, and k is a scalar, then

1. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
2. $(k\mathbf{a}) \times \mathbf{b} = k(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (k\mathbf{b})$
3. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
4. $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$
5. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$
6. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

The area of the parallelogram determined by **a** and **b** is $|\mathbf{a} \times \mathbf{b}|$, the magnitude of the cross product between the vectors.

The volume of the parallelepiped determined by **a**, **b** and **c** is $|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$. Note that here, the bars denote absolute value rather than magnitude since the dot product gives a scalar.

§**12.5** Equations of Lines and Planes

We can represent a line through point $P_0(x_0, y_0, z_0)$ with direction vector $\mathbf{v} = \langle a, b, c\rangle$ by starting at the origin, traveling along the position vector $\mathbf{r}_0 = \langle x_0, y_0, z_0\rangle$ of P_0, and adding any scalar multiple t of the direction vector. This gives the **vector equation** of the line

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$$

The corresponding **parametric equations** are

$$x = x_0 + at \qquad y = y_0 + bt \qquad z = z_0 + ct$$

Any two distinct lines can intersect at exactly one point, be parallel to each other, or be **skew** which is what we call non-intersecting and non-parallel lines.

We can express a *line segment* between two points $P_0(x_0, y_0, z_0)$ and $P_1(x_1, y_1, z_1)$ by finding an equation for the line connecting the two points and restricting t.

$$\mathbf{r}(t) = (1-t)\mathbf{r}_0 + t\mathbf{r}_1 \qquad 0 \le t \le 1$$

Since planes contain infinitely many vectors in different directions, we identify a plane by a point $P_0(x_0, y_0, z_0)$ in the plane and a vector $\mathbf{n} = \langle a, b, c\rangle$ that is orthogonal to the plane. A vector orthogonal to the plane is called a **normal vector**. As with lines, any nonzero scalar multiple of the normal vector can also be used to write the equation. For any point $P(x, y, z)$ in the plane, the vector $\overrightarrow{P_0P}$ lies in the plane, so we know $\mathbf{n}$ is orthogonal to $\overrightarrow{P_0P}$ which gives the **vector equation** for a plane

$$\mathbf{n} \cdot \overrightarrow{P_0P} = 0$$

Substituting the values, we get the **linear equation**

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Given any two planes, we can use the normal vectors to describe the interactions and intersections between the planes. As with two lines in two-dimensional space, two planes in three-dimensional space will either intersect (in a line rather than a point) or be parallel.

§**12.6** Cylinders and Quadric Surfaces

Cross-sections or **traces** of a surface are curves formed by the intersection of the surface with planes parallel to the coordinate planes. Graphing these curves helps us sketch the graph of a figure in three-dimensional space.

A **cylinder** is a surface that consists of all lines that are parallel to a given line and pass through a given curve. Typically, the equation of a cylinder relates two variables and the third variable can be anything.

The equation of a **quadric surface** has all three variables and two or more variables are squared.

The surfaces and quadric equations that you should know are in the table below. See the book or your handout for examples of each type.

Surface	**Cross-Sections**
Circular Cylinder	Circles in one direction; Lines in two directions
Parabolic Cylinder	Parabolas in one direction; Lines in two directions
Ellipsoid	Ellipses in all directions
Elliptic Paraboloid	Ellipses in one direction; Parabolas in two directions;
Hyperbolic Paraboloid	Hyperbolas in one direction; Parabolas in two directions
Cone	Ellipses in one direction (radius 0 at some point); Hyperbolas in two directions
Hyperboloid of One Sheet	Ellipses in one direction (radius never equals 0); Hyperbolas in two directions
Hyperboloid of Two Sheets	Ellipses in one direction (radius is "negative" somewhere, so no ellipses); Hyperbolas in two directions

Chapter 13: Vector-Valued Functions

§**13.1** Vector-Valued Functions and Space Curves

A **vector function** is a function that takes real numbers (its domain) to vectors (its range). We can write

$$\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle = f(t)\hat{\mathbf{i}} + g(t)\hat{\mathbf{j}} + h(t)\hat{\mathbf{k}}$$

where $f(t), g(t)$, and $h(t)$ are the **component functions** of **r**.

There are three main instances where domain issues occur (meaning the function is undefined). They are

1. Division by 0
2. A negative number under an even root
3. A negative or 0 inside a logarithmic function

The domain is all possible values of t that avoid these potential issues.

If one particle has position function $\mathbf{r}_0(t) = \langle f_0(t), g_0(t), h_0(t)\rangle$ and another particle has position function $\mathbf{r}_1(t) = \langle f_1(t), g_1(t), h_1(t)\rangle$, we can find out if their paths intersect by solving the system

$$\begin{cases} f_0(t) = f_1(s) \\ g_0(t) = g_1(s) \\ h_0(t) = h_1(s) \end{cases}$$

If there exists values for t and s that satisfy those three equations, the paths intersect at some point. If t and s equal the same value there, the particles collide. We can also figure out that the particles collide by assuming $s = t$ and solving *this* system

$$\begin{cases} f_0(t) = f_1(t) \\ g_0(t) = g_1(t) \\ h_0(t) = h_1(t) \end{cases}$$

If there exists a t that satisfies this system, then the particles collide because they are at the same point at the same time.

We can relate a vector function to a function of x, y, and z by setting $x = f(t)$, $y = g(t)$, and $z = h(t)$.

§**13.2** Derivatives and Integrals of Vector Functions

If $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$, then the derivative is

$$\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t)\rangle$$

The indefinite integral is

$$\int \mathbf{r}(t)dt = \left\langle \int f(t)dt, \int g(t)dt, \int h(t)dt \right\rangle + \mathbf{C}$$

where $\mathbf{C} = \langle c_1, c_2, c_3 \rangle$ is a vector of constants which can be found using initial data.

The definite integral is

$$\int_a^b \mathbf{r}(t)dt = \left\langle \int_a^b f(t)dt, \int_a^b g(t)dt, \int_a^b h(t)dt \right\rangle$$

or if $\mathbf{R}(t) = \int \mathbf{r}(t)dt$, the the fundamental theorem of calculus gives $\int_a^b \mathbf{r}(t)dt = \mathbf{R}(b) - \mathbf{R}(a)$.

Differentiation Rules

Suppose **u**(t) and **v**(t) are differentiable vector functions, c is a scalar, and f is a real-valued function. Then

1. $\frac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t)$
2. $\frac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$
3. $\frac{d}{dt}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$
4. $\frac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)] = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$
5. $\frac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$
6. $\frac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t))$

Similar to the two-dimensional case, $\mathbf{r}'(t)$ is the tangent vector at the point $P(f(t), g(t), h(t))$ on the curve $\mathbf{r}(t)$. The tangent vector gives the rate of change in each direction with respect to t.

§**13.3** Arc Length and Curvature

The **length of a curve** $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$ with $a \le t \le b$ is

$$L = \int_a^b |\mathbf{r}'(t)|dt = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}dt$$

or

$$L = \int_a^b \sqrt{\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2 + \left[\frac{dz}{dt}\right]^2}dt$$

We define the arc length function as the length of the curve $\mathbf{r}(\tau)$ from a value $\tau = a$ to $\tau = t$ by the integral

$$s(t) = \int_a^t |\mathbf{r}'(\tau)|d\tau$$

which gives us $\frac{ds}{dt} = |\mathbf{r}'(t)|$.

We can reparameterize a curve $\mathbf{r}(t)$ with respect to arc length by finding $s(t)$ then solving for t in terms of s. After that, we find $\mathbf{r}(t(s)) = \mathbf{r}(s)$.

The **unit tangent vector** is given by

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

The **unit normal vector** is orthogonal to the unit tangent vector and is given by

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$$

The **curvature** is a measure of how quickly the curve changes direction at a point and is given by

$$\kappa(s) = \left|\frac{d\mathbf{T}}{ds}\right| \qquad \kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}$$

We mostly use the equation on the right.

§**13.4** Motion in Space: Velocity and Acceleration

Consider a position function $\mathbf{r}(t)$.

We can find the **exact** velocity and acceleration using derivatives.

$$\mathbf{v}(t) = \mathbf{r}'(t) \qquad \mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$$

The speed is

$$|\mathbf{v}(t)| = |\mathbf{r}'(t)| = \frac{ds}{dt}$$

Chapter 14: Partial Derivatives

§**14.1** Functions of Several Variables

A function of several variables takes a point (for example (x, y), (x, y, z), (x, y, t), etc.) and maps it to a number. The domain of the function are all valid points. We will mostly talk about the domain of a function of two variables where we can map the domain in the xy-plane. The same domain issues as in §13.1 could arise. Make sure when you graph a domain, you use a solid line if values on the line are in the domain and you use a dashed line if values on the line are NOT in the domain. Then shade the region where all the points in the domain lie.

For a function of two variables, we usually consider $z = f(x, y)$. In this case, functions of the form $f(x, y) = ax + by + c$ are called **linear** where a, b, and c are scalars. In three-dimensional space, the graph of a linear function is a plane.

To help draw the graph of a function, we use **level curves** (or **contour lines**) which are the curves $f(x, y) = k$ with k a constant. To make a **contour map**, we plot several (three or more) level curves on one graph.

§**14.2** Limits and Continuity

If f is a function of two variables, we write the limit of $f(x, y)$ as (x, y) approaches (a, b) as

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

When (a, b) and points near (a, b) are in the domain of $f(x, y)$, then we can evaluate the limit by finding $f(a, b)$.

If (a, b) is not in the domain, and we get $f(a, b) = \frac{0}{0}$, then first, we can try to rewrite $f(x, y)$ by factoring or rationalizing $f(x, y)$ and evaluate again. If that still gives us $\frac{0}{0}$, then we try to approach (a, b) along different curves. The most useful ones will be $y = mx$, $y = mx^2$, and $x = my^2$. In this case, we can substitute into the function and get a limit in terms of one variable only.

If this limit equals a function of m such as $\frac{m}{1-4m^2}$, then the limit **does not exist** because m is arbitrary, so depending on which line we use, we get a different limit.

If the limit doesn't depend on m for one of the functions, then we can choose a different one of the above functions and it will probably equal a different limit.

In this course, these are the only types of limits we will encounter. However, if a limit does not fit the above, we could resort to graphing or numerical methods to find the limit.

A function is **continuous** at a point (a, b) if

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$$

.

We can find where a function is not continuous by finding the points not in the domain of f. In this course, usually when a point is in the domain, the function is continuous at that point, and furthermore, it will usually be differentiable at the point (meaning we can take the derivative at that point).

§**14.3** Partial Derivatives

When we have a function of two or more variables, we have to be careful when we take the derivative because we can take the derivative with respect to any variable. Once we decide with respect to which variable we're going to take the derivative, we treat the other variables as constants. For instance if $z = x^2y$, $\frac{\partial z}{\partial x} = 2xy$ and $\frac{\partial z}{\partial y} = x^2$.
Some of the different notation we can use for partial derivatives of $z = f(x, y)$ are

$$f_x(x,y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x,y) = \frac{\partial z}{\partial x} = z_x$$

$$f_y(x,y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x,y) = \frac{\partial z}{\partial y} = z_y$$

We can also find second (or third or fourth) partial derivatives by taking the derivative with respect to the same variable twice or with respect to two different variables. Almost always, in this course, we will have that $f_{xy} = f_{yx}$.

§**14.4** Tangent Planes and Linear Approximations

Similar to how we found the tangent lines in §13.2, an equation of the tangent plane to the surface $F(x, y, z) = 0$ at the point $P(x_0, y_0, z_0)$ is

$$F_x(x_0,y_0,z_0)(x - x_0) + F_y(x_0,y_0,z_0)(y - y_0) + F_z(x_0,y_0,z_0)(z - z_0) = 0$$

If we can write $z = f(x, y)$ we can write the tangent plane as $z = z_0 + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$. We can use this equation to approximate values on surface at a different point. This gives the **linear approximation** (because the equation of a plane is linear) of f at (a, b)

$$f(x,y) \approx L(x,y) = f(a,b) + f_x(a,b)(x - a) + f_y(a,b)(y - b)$$

If we know the function, we can find the **increment** which is the exact change in z when (x, y) changes

$$\Delta z = f(a + \Delta x, b + \Delta y) - f(a,b)$$

The **total differential** dz is the change in z when we use our tangent plane approximation

$$dz = f_x(x,y)dx + f_y(x,y)dy = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

If we take $dx = \Delta x = x - a$ and $dy = \Delta y = y - b$. we get

$$dz = f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

§**14.5** The Chain Rule

Suppose $z = f(x,y)$ where $x = g(t)$ and $y = h(t)$, then we could write z as a function of t. However, this function can get very complicated, so we often leave it in terms of x and y. Then to take the derivative with respect to t, we have

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

If instead, we have $x = g(s,t)$ and $y = h(s,t)$, then we could write z as a function of s and t, but again, that can get really complicated. Now, if we want to take the derivative of z with respect to s or t, we have

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

We can also use partial derivatives to find $\frac{dy}{dx}$ in an equation with only x and y for variables. To do this, we move all terms to one side of the equation so that we have $F(x,y) = 0$. Then we have

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F_x}{F_y}$$

§**14.6** Directional Derivatives and the Gradient Vector

The **directional derivative** of f at (x_0, y_0) in the direction of a *unit vector* $\mathbf{u} = \langle a, b\rangle$ is

$$D_{\mathbf{u}} f(x_0,y_0) = f_x(x_0,y_0)a + f_y(x_0,y_0)b.$$

Note that $\mathbf{u}$ is a unit vector, so if you're looking for the directional derivative in the direction of an arbitrary vector $\mathbf{v}$, you need to find $\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|}$ before finding the directional derivative. The directional derivative is the rate of change of the function in a direction, so we want to use a unit vector because the magnitude of the change shouldn't be affected by length of the direction vector.

The **gradient** of a function f is

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle$$

or

$$\nabla f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)\rangle$$

depending on how many variables the function depends on. Now, we can write the directional derivative as

$$D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}.$$

Remember that we can write the dot product as $\nabla f \cdot \mathbf{u} = |\nabla f||\mathbf{u}| \cos(\theta)$, so the maximum value of the directional derivative at a point $\mathbf{x}$ is $|\nabla f(\mathbf{x})|$ when $\mathbf{u}$ is in the same direction as $\nabla f(\mathbf{x})$ meaning $\theta = 0$.