MA 261 Exam 2 Review

Chapter 15: Functions of Several Variables

§**15.7** Maximum/Minimum Problems

To find **local extrema** of a function $f(x, y)$, we first find the critical points by solving the system of equations

$$\begin{cases} f_x(x,y) = 0 \\ f_y(x,y) = 0 \end{cases}$$

Once we find the critical points, we use the Second Derivatives Test to classify each critical point as a local minimum, local maximum, or a saddle point. Let's focus on a general critical point (a, b). First we compute $D(a,b) = f_{xx}(a,b)f_{yy}(a,b) - [f_{xy}(a,b)]^2$.
If $D(a, b) < 0$, then f has a saddle point at (a, b).
If $D(a, b) > 0$, then $f(x, y)$ has a local minimum or maximum at (a, b).
We can figure out if it's a min or a max by using $f_{xx}(a, b)$.
If $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.
If $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.
If $D(a, b) = 0$, then the Second Derivative Test is inconclusive, and we have to use graphing or other means to classify the critical point.

If we are considering a function $f(x, y)$ on a closed and bounded domain D, then we can find the **absolute extrema** of f on D by doing the following:

1. Find the critical points of f in D.
2. Evaluate f at the critical points **and** on the boundary of D (NOT just the corners of the boundaries).
3. The largest value from step 2 is the absolute maximum, and the smallest value from step 2 is the absolute minimum. **Do NOT use the Second Derivative Test to classify the min and max!**

§**15.8** Lagrange Multipliers

We use Lagrange Multipliers when we want to find the minimum and maximum value of a function f subject to a constraint $g = k$. In many cases this is another method of finding absolute minimums and maximums on a closed and bounded domain as in the previous section. Either way, we follow this format:

1. Solve the system of equations $\begin{cases} \nabla f = \lambda \nabla g \\ g = k \end{cases}$
2. Evaluate f at all points found in step 1.

3. The largest value from step 2 is the maximum value, and the smallest value from step 2 is the minimum value.

The most difficult or confusing part of using Lagrange Multipliers comes from step 1 above. To solve the system of equations, we usually want to pick one of the equations from the gradient (for instance, $f_x = \lambda g_x$) and solve for λ. The we can substitute that equation, relation, or number for λ into one of the other equations from the gradient. The goal is to use the equations from the gradient to find relationships between the variables that can be substituted into the constraint to solve explicitly for one of the variables.
NEVER divide by 0 when solving. Always consider if a denominator can equal 0 to find ALL possible solutions.

Chapter 16: Multiple Integration

§**16.1** Double Integrals over Rectangular Regions

If $f(x,y) \geq 0$, then the volume of the solid that lies above the rectangle R and below the surface $z = f(x,y)$ is

$$V = \iint_R f(x,y)dA$$

The limits of integration will be $left \leq x \leq right$ and $bottom \leq y \leq top$
To evaluate a double integral over a rectangle $R = \{(x,y) \mid a \leq x \leq b, c \leq y \leq d\}$, we have

$$\iint_R f(x,y)dA = \int_a^b \int_c^d f(x,y)\ dy\ dx = \int_c^d \int_a^b f(x,y)\ dx\ dy$$

If we can write $f(x,y) = g(x)\ h(y)$, then we have

$$\iint_R f(x,y)dA = \int_a^b g(x)\ dx \int_c^d h(y)\ dy$$

§**16.2** Double Integrals over General Regions

When we integrate over a region that is not a rectangle, we have to think of the bounds as functions of variables. In general, we can write

$$D = \{(x,y) \mid a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$$

where $g_1(x)$ is the bottom function when the region of integration is graphed, and $g_2(x)$ is the top function to get

$$\iint_R f(x,y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\ dy\ dx$$

or we can write

$$D = \{(x,y) \mid h_1(y) \le x \le h_2(y), c \le y \le d\}$$

where $h_1(y)$ is the left function, and $h_2(y)$ is the top function to get

$$\iint_R f(x,y)\ dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\ dx\ dy$$

It is helpful to graph the region to make sure you pick the bounds correctly. Sometimes one order of integration will be much easier to evaluate than the other.

The **average value** of a function over a region in the xy-plane can be found with the formula

$$f_{ave} = \frac{1}{A(D)} \iint_D f(x,y)\ dA$$

where $A(D) = \iint_D\ dA$ is the area of the region. We can make sense of this formula by thinking about an object whose base is the shape of the region D. Then we integrate the function to find the volume of the object over the region D. The average value of f is the *uniform* height that the object would need to have to maintain its volume while have a flat top.

§**16.3** Double Integrals in Polar Coordinates

We can switch between rectangular coordinates (x, y) and polar coordinates (r, θ) using the formulas

$$r^2 = x^2 + y^2, \qquad \tan\theta = \frac{y}{x}, \qquad x = r\cos\theta, \qquad y = r\sin\theta$$

Then we can switch an integral from rectangular coordinates to polar coordinates

$$\iint_D f(x,y)\ dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\ \mathbf{r}\ dr\ d\theta$$

Do not forget $dA = \mathbf{r}\ dr\ d\theta$! The inserted r variable is often necessary to solve the integral.

§**16.4** Triple Integrals

We can integrate of function $f(x, y, z)$ of three variables over a solid E using a triple integral

$$\iiint_E f(x,y,z)\ dV$$

If we want to find the volume of the region E, we have

$$V(E) = \iiint_E dV$$

For triple integrals in rectangular coordinates, we can have dV equal to $dx\ dy\ dz$, $dx\ dz\ dy$, $dy\ dx\ dz$, $dy\ dz\ dx$, $dz\ dx\ dy$, or $dz\ dy\ dx$. For the limits of integration on the integrals, the outermost integral will **always** be explicit numbers, the middle integral can have functions of the outermost variable, and the innermost integral can have functions of the other two variables. Of course, it is always possible that all of the limits of integration are explicit numbers in which case, the order of integration can easily be switched. When the limits of integration are functions of the different variables, it is helpful to draw a picture to figure out how to reorder the integrals. I always think of the two outermost variables as the "area" while the innermost integral is of the "height." Then you can draw the relations between the two outer variables in a plane to switch their order.

§**16.5** Triple Integrals in Cylindrical and Spherical Coordinates

Cylindrical coordinates in three dimensions are given by (r, θ, z) where r and θ are polar coordinates in the xy-plane, and where z is as in rectangular coordinates. Note that we could also have cylindrical coordinates (x, r, θ) where r and θ are polar coordinates in the yz-plane or we could have (r, y, θ) where r and θ are polar coordinates in the xz-plane.

We can switch a triple integral from rectangular to cylindrical coordinates using

$$\iiint_E f(x, y, z)\ dV = \iiint_E f(r\cos\theta, r\sin\theta, z)\ \mathbf{r}\ dr\ d\theta\ dz$$

Again, we can do integration in any order, but we **must** have the **r** in the integral as with integrating double integrals in polar coordinates.

Cylindrical coordinates are most useful when dealing with a solid that is symmetric in a single plane (usually the xy-plane).

Spherical coordinates are given by (ρ, θ, φ) where ρ is the distance from the origin to the point, θ is measured counter clockwise from the positive x-axis, and φ is measured down from the positive z-axis with $0 \le \varphi \le \pi$. We can switch between rectangular and spherical coordinates using the formulas

$$x = \rho\sin\varphi\cos\theta \qquad y = \rho\sin\varphi\sin\theta \qquad z = \rho\cos\varphi \qquad \rho^2 = x^2 + y^2 + z^2$$

We can switch from a triple integral in rectangular coordinates to a triple integral in spherical coordinates by

$$\iiint_E f(x,y,z)\,dV = \iiint_E f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\ \rho^2\sin\varphi\ d\rho\ d\theta\ d\varphi$$

Notice that dV is more complicated than in cylindrical coordinates. Here, we have $\rho^2\sin\varphi$.

Spherical coordinates are most useful when dealing with a solid that is symmetric about the origin.

§**16.6** Integrals for Mass Calculations

Suppose a lamina with density $\rho(x,y)$ occupies a region D of the xy-plane. Then the **mass** of the lamina is

$$m = \iint_D \rho(x,y)\ dA$$

The **center of mass** is $(\bar{x},\bar{y})$, where

$$\bar{x} = \frac{1}{m}\iint_D \mathbf{x}\rho(x,y)\ dA \qquad \bar{y} = \frac{1}{m}\iint_D \mathbf{y}\rho(x,y)\ dA$$

We can find the center of mass $(\bar{x},\bar{y},\bar{z})$ of a solid with density $\rho(x,y,z)$ occupying a region of space E where

$$m = \iiint_E \rho(x,y,z)\ dV$$

$$\bar{x} = \frac{1}{m}\iiint_E \mathbf{x}\rho(x,y,z)\ dV \qquad \bar{y} = \frac{1}{m}\iiint_E \mathbf{y}\rho(x,y,z)\ dV$$

$$\bar{z} = \frac{1}{m}\iiint_E \mathbf{z}\rho(x,y,z)\ dV$$

Chapter 17: Vector Calculus

§**17.1** Vector Fields

A **vector field** assigns a vector ($\langle x,y\rangle$ or $\langle x,y,z\rangle$) to every point ((x,y) or (x,y,z)). **Gradient fields** are an example of a vector field. To sketch gradient fields, we pick points in the plane or in space, evaluate the vector field at the points, and use the point as the tail to draw the vector in the plane or in space.

If $\nabla f = \vec{F}$, we say the scalar-valued function f is the **potential function** of vector field $\vec{F}$.

§**17.2** Line Integrals

Recall from chapter 15 that the length of a curve with parameterization $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$ with $a \le t \le b$ is

$$L = \int_a^b |\vec{r}\,'(t)|\; dt = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\; dt$$

or

$$L = \int_a^b \sqrt{\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2 + \left[\frac{dz}{dt}\right]^2}\; dt$$

We define the arc length function as the length of the curve $\vec{r}(\tau)$ from a value $\tau = a$ to $\tau = t$ by the integral

$$s(t) = \int_a^t |\vec{r}\,'(\tau)|\; d\tau$$

which gives us $\frac{ds}{dt} = |\vec{r}\,'(t)|$, so $ds = |\vec{r}\,'(t)|\; dt$

Now, if we want to integrate a function along a curve C, we first parameterize the curve. There are a few different ways to parameterize a curve.

To parameterize a circle with radius r, we have $x = r\cos(t),\;\; y = r\sin(t),\;\; 0 \le t \le 2\pi$. If the curve is not the entire circle, we can change the limits on t.

To parameterize a curve $y = f(x)$ from $(a, f(a))$ to $(b, f(b))$, we can set $x = t$ and $y = f(t)$ with $a \le t \le b$.

We can parameterize a line segment between the points P_0 and P_1 with position vectors $\vec{r}_0$ and $\vec{r}_1$ as $\vec{r}(t) = (1-t)\;\vec{r}_0 + t\;\vec{r}_1,\;\; 0 \le t \le 1$.

Once we have the curve parameterized as

$$x = x(t), \qquad y = y(t), \qquad a \le t \le b$$

then we can evaluate the integral

$$\int_C f(x,y)\; ds = \int_a^b f(x(t), y(t))\sqrt{(x'(t))^2 + (y'(t))^2}\; dt$$

Note that ds refers to integration with respect to arc length as above and we have

$$ds = |\vec{r}\,'(t)|\; dt = \sqrt{(x'(t))^2 + (y'(t))^2}\; dt$$

Line integrals can also be given as integration with respect to x or y. If x and y are functions of t, then $dx = x'(t)\; dt$ and $dy = y'(t)\; dt$, and

$$\int_C f(x,y)\; dx = \int_a^b f(x(t), y(t))\; x'(t)\; dt$$

$$\int_C f(x,y)\,dy = \int_a^b f(x(t),y(t))\,y'(t)\,dt$$

When we deal with line integrals in space, we have an extra z variable, but the same basic equations will hold. For instance, we have $ds = |\vec{r}\,'(t)|\,dt = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\,dt$.

We can also compute line integrals of vector fields along a curve C by

$$\int_C \vec{F}\cdot\vec{T}\,ds = \int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt$$

Note that we take the dot product to compute this line integral so that we integrate a scalar function instead of a vector function.
If C is a closed curve, then $\int_C \vec{F}\cdot\vec{T}\,ds$ is called the **circulation**.

To find the **work** done by a force field to move an object along a curve, compute $\int_C \vec{F}\cdot d\vec{r}$.

To find the **flux** of a vector field through a simple *closed* curve, compute

$$\int_C \vec{F}\cdot\vec{n}\,ds = \int_C f\,dy - g\,dx$$