MA261 Final Exam Review

Chapter 17: Vector Calculus

§**17.1** Vector Fields

A **vector field** takes a point ((x, y) or (x, y, z)) as input and has output that is a vector ($\langle x, y\rangle$ or $\langle x, y, z\rangle$). We have already seen **gradient fields** which are an example of a vector field. To sketch gradient fields, we pick points in the plane or in space, evaluate the vector field at the points, and use the point as the tail to draw the vector in the plane or in space.

If $\nabla\varphi = \vec{F}$, we say $\vec{F}$ is a **gradient field**, and φ is its **potential function**.

§**17.2** Line Integrals

Recall from chapter 14 that the length of a curve $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$ with $a \le t \le b$ is

$$L = \int_a^b |\vec{r}\,'(t)|\, dt = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\, dt$$

or

$$L = \int_a^b \sqrt{\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2 + \left[\frac{dz}{dt}\right]^2}\, dt$$

We defined the arc length function as the length of the curve $\vec{r}(\tau)$ from a value $\tau = a$ to $\tau = t$ by the integral

$$s(t) = \int_a^t |\vec{r}\,'(\tau)|\, d\tau$$

which gives us $\frac{ds}{dt} = |\vec{r}\,'(t)|$, so $ds = |\vec{r}\,'(t)|\, dt$

If we want to integrate a function along a curve C, we first parameterize the curve. There are many different ways to parameterize a curve. Parameterization means that we relate all coordinates (x and y or $x, y,$ and z) to a single parameter (usually t).

To parameterize a circle with radius r, we have $x = r\cos(t),\;\; y = r\sin(t),\;\; 0 \le t \le 2\pi$. If the curve is not the entire circle, we can change the limits on t.

To parameterize a curve $y = f(x)$ from $(a, f(a))$ to $(b, f(b))$, we can set $x = t$ and $y = f(t)$ with $a \le t \le b$.

We can parameterize a line segment between the points P_0 and P_1 with position vectors $\vec{r}_0$ and $\vec{r}_1$ as $\vec{r}(t) = (1-t)\,\vec{r}_0 + t\,\vec{r}_1,\;\; 0 \le t \le 1$.

Once we have the curve parameterized as

$$x = x(t), \qquad y = y(t), \qquad a \le t \le b$$

then we can evaluate the integral

$$\int_C f(x,y)\ ds = \int_a^b f(x(t),y(t))\sqrt{(x'(t))^2+(y'(t))^2}\ dt$$

Note that ds refers to integration with respect to arc length as above and we have

$$ds = |\vec{r}\,'(t)|\ dt = \sqrt{(x'(t))^2+(y'(t))^2}\ dt$$

Line integrals can also be given as integration with respect to x or y. If x and y are functions of t, then $dx = x'(t)\ dt$ and $dy = y'(t)\ dt$, and

$$\int_C f(x,y)\ dx = \int_a^b f(x(t),y(t))\ x'(t)\ dt$$

$$\int_C f(x,y)\ dy = \int_a^b f(x(t),y(t))\ y'(t)\ dt$$

When we deal with line integrals in space, we have an extra z variable, but the same basic equations will hold. For instance, we have $ds = |\vec{r}\,'(t)|\ dt = \sqrt{(x'(t))^2+(y'(t))^2+(z'(t))^2}\ dt$.

Notice that if we trace the curve C in the **opposite** direction, t goes from b to a instead of a to b. We call this curve $-C$. For the integral, we have

$$\int_{-C} \vec{F}\cdot d\vec{r} = \int_b^a \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\ dt = -\int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\ dt = -\int_C \vec{F}\cdot d\vec{r}$$

In short, we have

$$\int_C \vec{F}\cdot d\vec{r} = -\int_{-C} \vec{F}\cdot d\vec{r}$$

We can also compute line integrals of vector fields along a curve by

$$\int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\ dt$$

Note that we take the dot product to compute this line integral so that we integrate a scalar function instead of a vector function.

Work done by a force field $\vec{F}$ to move an object along a curve C in the positive direction is

$$w = \int_C \vec{F}\cdot\vec{T}\ ds$$

If C is a closed curve, then $\int_C \vec{F}\cdot\ d\vec{r}$ is called the **circulation**.

The **flux** of vector field $\vec{F}$ along a curve is computed

$$\int_C \vec{F}\cdot\vec{n}\,ds$$

Normally, in this course, we talk about flux through a closed curve but C does NOT have to be closed.

§**17.3** Conservative Vector Fields

If a vector field $\vec{F}=\langle f,\ g\rangle$ is **conservative**, then

$$\frac{\partial f}{\partial y}=\frac{\partial g}{\partial x}$$

If a vector field $\vec{F}=\langle f,\ g,\ h\rangle$ is **conservative**, then

$$\frac{\partial f}{\partial y}=\frac{\partial g}{\partial x},\quad \frac{\partial f}{\partial z}=\frac{\partial h}{\partial x},\quad \text{and}\quad \frac{\partial g}{\partial z}=\frac{\partial h}{\partial y}$$

If a vector field is conservative, then we can find a potential function φ such that $\vec{F}=\nabla\varphi$. Remember that when integrating with respect to one variable, the constant of integration added to the function is not a constant scalar or vector. The constant will be a constant function of the *other* variables.

If $\vec{F}$ is a conservative field with $\vec{F}=\nabla\varphi$ and C is a curve (or contour) given by $\vec{r}(t), a\le t\le b$, then the Fundamental Theorem for Line Integrals says

$$\int_C \vec{F}\cdot d\vec{r}=\varphi(\vec{r}(b))-\varphi(\vec{r}(a))$$

If we know the curve begins at point A and ends at point B, we have

$$\int_C \vec{F}\cdot d\vec{r}=\varphi(B)-\varphi(A)$$

We can deduce two things from this. The first thing is that for a conservative vector field, the value of $\int_C \vec{F}\cdot d\vec{r}$ does not depend on the parameterization (or path) of the curve C. It only depends on the endpoints of the curve. (We say the curve is "path independent.") The second thing we see is that if C is a closed path, then $A=B$, so $\varphi(B)=\varphi(A)$ which means $\int_C \vec{F}\cdot d\vec{r}=\varphi(B)-\varphi(A)=0$.

§**17.4** Green's Theorem

For a **closed curve** C, the curve has **positive orientation** if it the interior of the curve (the section being bounded) is to the left when traversing the curve. Normally, this occurs for counterclockwise traversal. To indicate that an integral is taken over a closed curve, we use $\oint$ instead of $\int$.

The **circulation form of Green's Theorem** states that if C is a **positively oriented** and **closed** curve that bounds a region R, then

$$\oint_C \vec{F} \cdot d\vec{r} = \int_C f\ dx + g\ dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\ dA$$

Note that if $\vec{F} = \langle f, g \rangle$ is a conservative vector field, $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$, so we see the integral is 0. We can also see this using the Fundamental Theorem for Line Integrals.

The **flux form of Green's Theorem** states that if C is a **positively oriented** and **closed** curve that bounds a region R, then

$$\oint_C \vec{F} \cdot d\vec{r} = \int_C f\ dy - g\ dx = \iint_R \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right)\ dA$$

We can also use a line integral to find the area of a region R enclosed by a curve C three ways:

$$\begin{aligned} A &= \frac{1}{2}\oint_C x\ dy - y\ dx \\ &= \oint_C x\ dy \\ &= -\oint_C y\ dx \end{aligned}$$

§**17.5** Curl and Divergence

Consider a vector field $\vec{F} = \langle f, g, h \rangle$ in three-dimensional space.

First, we have **del**, the vector differential operator, given by

$$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$$

Note that if we apply the del operator to a scalar field $f(x, y, z)$, we get the gradient

$$\text{grad } f = \nabla f$$

Now, we can define the **curl** of a vector field in terms of the del operator

$$\text{curl } \vec{F} = \nabla \times \vec{F}$$

For the three-dimensional vector fields $\vec{F}$ that we'll encounter, if curl $\vec{F} = 0$, then $\vec{F}$ is a conservative vector field.

Finally, we have the **divergence** of a vector field $\vec{F}$ given by

$$\text{div } \vec{F} = \nabla \cdot \vec{F}$$

Using these definitions, we can see that div $\left(\text{curl } \vec{F}\right) = 0$.

Note that we apply the del operator to a scalar field and get a vector field; we take the curl of a vector field and get a vector field; and we take the divergence of a vector field and get a scalar field.

§**17.6** Surface Integrals

We described a curve in space by a vector function in terms of of a single variable t, i.e. $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$. We can describe a **surface** by a vector function of two parameters u and v, i.e. $\vec{r}(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle$. Note that the domain of this function is in the uv-plane, but we draw the surface in space with x, y, and z coordinates.

The most useful way to recognize a surface in the form of a vector equation is using $x = a\cos\theta$ and $y = b\sin\theta$ or rather, $\frac{x}{a} = \cos\theta$ and $\frac{y}{b} = \sin\theta$. Then we know $1 = \cos^2\theta + \sin^2\theta = \frac{x^2}{a^2} + \frac{y^2}{b^2}$. Remember that we can rotate the axes to use the equalities in the xy-, xz-, or yz-planes. Recalling cylindrical and spherical coordinates will also be helpful.

We can also think of **grid curves/lines** where one of the variables in fixed while the other can change. It's similar to considering the cross-sections of three-dimensional solids to determine their shapes.

Using the vector representation $\vec{r}(u, v)$ of a surface, we can find the **tangent plane** to the surface at a point $P_0 = (x_0, y_0, z_0)$ with position vector $\vec{r}(u_0, v_0) = \langle x_0, y_0, z_0\rangle$ using the cross product. If $\vec{r}_u = \frac{\partial \vec{r}}{\partial u}$ and $\vec{r}_v = \frac{\partial \vec{r}}{\partial v}$, then the normal vector to the tangent plane *at any point* is

$$\vec{n}\ (u, v) = \vec{r}_u \times\ \vec{r}_v$$

Then $\vec{n}\ (u_0, v_0) = \langle a, b, c\rangle$ is a vector normal to the plane at the point P_0. Now, we have the usual equation for a tangent plane

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Just as we integrated over areas, volumes, and lines, we can integrate over surfaces using $dS = |\vec{r}_u \times \vec{r}_v|\ dA$. We get that the surface integral of a *scalar* function f over a surface S given by the parametric equation $\vec{r}(u, v)$ is

$$\iint_S f(x, y, z)\ dS = \iint_R f(\vec{r}(u, v))\ |\vec{r}_u \times \vec{r}_v|\ dA$$

Note that in this instance, $dA = du\ dv$ or $dA = dv\ du$, but we can also switch to polar coordinates if it will make the integration easier. Often, we will need to separate the surface into different parts to parameterize them easily. When we do this, we just add the separate integrals together.

We can find the **area of a surface** by computing

$$\iint_S\ dS = \iint_R |\vec{r}_u \times \vec{r}_v|\ dA$$

If we can **explicitly define** a surface $z = g(x, y)$ by the parameterization $\vec{r}(x, y) = \langle x, y, g(x, y)\rangle$, then we get $|\vec{r}_x \times \vec{r}_y| = \sqrt{1 + (g_x)^2 + (g_y)^2}$. This means that

$$A(S) = \iint_R \sqrt{1 + (g_x)^2 + (g_y)^2}\ dA$$

We call S an **oriented surface** when we choose a normal vector $\vec{n}$ to describe the orientation of the surface. We have two normal vectors to choose from $\vec{n_1} = \vec{r}_u \times \vec{r}_v$ or $\vec{n_2} = -\vec{n_1}$. We can also use the unit normal vector $\hat{\mathbf{n}} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$.

For any surface, we describe the upward orientation as a normal vector with a positive z-component and downnward orientation as a normal vector with a negative z-component. For a **closed** surface, the **positive orientation** is the one that has normal vectors pointing *outward*, and negative orientation points inward.

Now that surfaces can be described in some sense with magnitude and direction, we can take surface integrals over vector fields (similar to taking line integrals over vector fields in section 2). The **flux** of $\vec{F}$ across S or more specifically, the surface integral of the vector field $\vec{F}$ over an oriented surface S with *unit* normal vector $\hat{\mathbf{n}}$ is

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \hat{\mathbf{n}}\ dS$$

Note that we are replacing $d\vec{S}$ with $\hat{\mathbf{n}}\ dS$. One way to think about this is that instead of considering the orientation of S, we only consider the parts of $\vec{F}$ that are normal to the oriented surface. Now we are just considering a surface integral. We know that $dS = |\vec{r}_u \times \vec{r}_v|\ dA$ and $\hat{\mathbf{n}} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$, so

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F} \cdot (\vec{r}_u \times \vec{r}_v)\ dA$$

§**17.7** Stokes' Theorem

Stokes' Theorem is a higher-dimensional version of Green's Theorem. Green's theorem is the special case where a surface lies completely in a plane, and we have a closed curve *in the plane* bounding a region *in the plane*. Now, we are going to have a **boundary curve** in space. Think of a boundary curve of a surface like the rim on a glass. The glass itself would be the surface, but the surface stops at the rim, so the rim bounds it. When we evaluate along a space curve, we are evaluating a line integral.

Stokes' Theorem states that if S is an *oriented* surface bounded by a *closed* and *positively oriented* curve C and if $\vec{F}$ is a vector field, then

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl } \vec{F} \cdot d\vec{S}$$

Since the curl of $\vec{F}$ is a vector field, we can evaluate the right side of the equation as an oriented surface integral as in section 6. Specifically, we can parameterize the surface S with $\vec{r}(u, v)$, then $\iint_S \text{curl } \vec{F} \cdot d\vec{S} = \iint_D \text{curl } \vec{F} \cdot (\vec{r}_u \times \vec{r}_v)\ dA$

Since the left side of the equation is a line integral of a vector field, we can use the techniques from section 2 or the Fundamental Theorem of Line Integrals. Specifically, we can parameterize the boundary curve C with $\vec{r}(t), a \le t \le b$, then $\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\ dt$.

Suppose $\vec{F}$ is conservative. For the left side of the equation, we are integrating around a closed curve, so the Fundamental Theorem of Line Integrals shows that the line integral is 0. For the right side of the equation, $\vec{F}$ is conservative, curl $\vec{F} = 0$, which shows that the surface integral is 0.

§**17.8** The Divergence Theorem

The **Divergence Theorem** states that if D is a *solid region* and S is the *positively* (i.e. outward) oriented boundary *surface* of E and if $\vec{F}$ is a vector field, then

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \text{div } \vec{F}\ dV$$

This theorem is especially useful when $\vec{F}$ has very complicated components or when E is a solid whose volume is easily computed.

If we have a hollow region D with inside boundary S_1 and outside boundary S_2, then the Divergence Theorem gives

$$\begin{aligned}\iiint_E \text{div } \vec{F}\, dV &= \iint_S \vec{F} \cdot d\vec{S} \\ &= \iint_{S_2} \vec{F} \cdot d\vec{S} - \iint_{S_1} \vec{F} \cdot d\vec{S}\end{aligned}$$