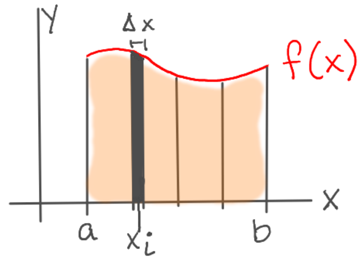
§16.1 Double Integrals over Rectangular Regions

Single variable function:



$A = \int_a^b f(x)\,dx$ is the area under the curve on the interval $[a, b]$

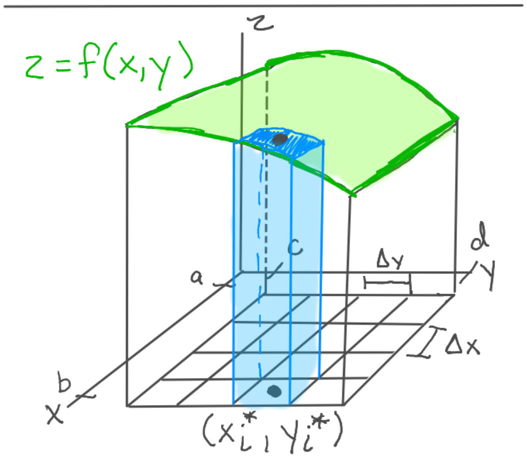
We can estimate the area with Riemann sums:

$A \approx \sum_{i=1}^{n} f(x_i)\,\Delta x$, where $f(x_i)$ is the height and Δx is the width of the rectangle.

As we increase the number of rectangles to estimate the area, we get better estimates.

We write $A = \int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$

Several variable function:



When we integrate a surface, we get the volume under the surface instead of area.

Instead of estimating the area of rectangles, we estimate the volume of rectangular prisms.

$$V \approx \sum_{i=1}^{n}\sum_{j=1}^{m} f(x_i^*, y_j^*)\underbrace{\Delta y\,\Delta x}_{=\Delta A} = \sum_{j=1}^{m}\sum_{i=1}^{n} f(x_i^*, y_j^*)\underbrace{\Delta x\,\Delta y}_{=\Delta A}$$

Again $f(x_i^*, y_j^*)$ is the height, but now we multiply by the area of the base.

When we let n and m increase infinitely:

$$V = \iint_R f(x,y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{m} f(x_i^*, y_j^*)\,\Delta A$$

where R is the rectangle $[a,b] \times [c,d]$.

$dA = dx\,dy$

$$V = \int_{y=c}^{y=d}\int_{x=a}^{x=b} f(x,y)\,dx\,dy$$

$dA = dy\,dx$

$$V = \int_{x=a}^{x=b}\int_{y=c}^{y=d} f(x,y)\,dy\,dx$$

This idea extends to any number of variables. For a function of 3 variables $f(x,y,z)$, we usually think of density or temperature. We write the general integral as $\iiint_D f(x,y,z)\, dV$

Ex.1 Evaluate $\iint_R x^2 y\, dA$ where $R = \{(x,y) \mid 0 \le x \le 1,\ -1 \le y \le 2\}$.

We can evaluate this three different ways:

① $$\underbrace{\int_{-1}^{2} \underbrace{\int_0^1 x^2 y\, dx}_{\text{limits on } x}\, dy}_{\text{limits on } y} = \int_{-1}^{2} \left[\int_0^1 x^2 y\, dx \right] dy$$

Integrate with respect to x, treating y as a constant

$$= \int_{-1}^{2} \left[\frac{x^3}{3} \cdot y \right]_{x=0}^{x=1} dy$$

$$= \int_{-1}^{2} \left[\frac{1}{3} \cdot y - \frac{0}{3} \cdot y \right] dy$$

$$= \int_{-1}^{2} \frac{1}{3} y\, dy$$

Only y should remain in the function!

$$= \frac{1}{3} \cdot \frac{1}{2} y^2 \Big|_{y=-1}^{y=2}$$

$$= \frac{1}{6} y^2 \Big|_{-1}^{2} = \frac{4}{6} - \frac{1}{6} = \boxed{\frac{1}{2}}$$

② $$\int_0^1 \int_{-1}^{2} x^2 y\, dy\, dx = \int_0^1 \left[\int_{-1}^{2} x^2 y\, dy \right] dx$$

Integrate with respect to y, treating x as a constant.

$$= \int_0^1 \left[\frac{1}{2} x^2 y^2 \right]_{y=-1}^{y=2} dx$$

$$= \int_0^1 \left[\frac{1}{2} \cdot x^2 \cdot 2^2 - \frac{1}{2} x^2 \cdot (-1)^2 \right] dx$$

$$= \int_0^1 \left[2x^2 - \frac{1}{2} x^2 \right] dx$$

$$= \int_0^1 \frac{3}{2} x^2\, dx$$

$$= \frac{1}{2} x^3 \Big|_{x=0}^{x=1} = \boxed{\frac{1}{2}}$$

③ $$\int_{-1}^{2} \int_0^1 \underbrace{x^2 y}_{\substack{\text{constant with}\\ \text{respect to } x\text{, so}\\ \text{factor out}}}\, dx\, dy = \int_{-1}^{2} y \left[\int_0^1 x^2\, dx \right] dy = \left[\int_0^1 x^2\, dx \right] \left[\int_{-1}^{2} y\, dy \right]$$

constant with respect to y, so factor out

$$= \left[\frac{1}{3}x^3\right]_{x=0}^{x=1} \cdot \left[\frac{1}{2}y^2\right]_{y=-1}^{y=2}$$
$$= \frac{1}{3} \cdot \frac{3}{2} = \boxed{\frac{1}{2}}$$

Note: We can only integrate this way if we can separate the variables with multiplication, i.e. $f(x,y) = g(x) \cdot h(y)$

Ex.2 Evaluate $\int_1^2 \int_0^1 (3x^2 + 4y^3)\,dy\,dx$.

Note: We cannot separate these integrals as in 3 above.

$$\int_1^2 \Big[\int_0^1 (\underbrace{3x^2}_{\text{constant with respect to } y} + 4y^3)\,dy\Big]\,dx = \int_1^2 \left[3x^2 y + y^4\right]_{y=0}^{y=1} dx$$
$$= \int_1^2 [3x^2 + 1]\,dx$$
$$= x^3 + x \Big|_{x=1}^{x=2}$$
$$= 8 + 2 - (1+1) = \boxed{8}$$

Ex.3 Evaluate $\int_1^2 \int_0^{\pi/2} xy\cos(x^2 y)\,dy\,dx$.

If we try to integrate as written, we need to do integration by parts: $u = xy$, $du = x\,dy$; $dv = \cos(x^2 y)\,dy$, $v = \frac{1}{x^2}\sin(x^2 y)$

Remember, x is constant with respect to y, so this is the same as integrating $\int \cos(4y)\,dy = \frac{1}{4}\sin(4y)$.

What happens if we switch the order of integration?

$\int_0^{\pi/2} \int_1^2 xy\cos(x^2 y)\,dx\,dy \rightarrow$ Now, we can use u-sub to integrate: $u = x^2 y$, $du = 2xy\,dx$

$$= \int_0^{\frac{\pi}{2}} \left[\int_{x=1}^{x=2} \frac{1}{2}\cos(u)\,du\right] dy$$
$$= \int_0^{\pi/2} \left[\frac{1}{2}\sin(x^2 y)\right]_{x=1}^{x=2} dy$$
$$= \frac{1}{2}\int_0^{\pi/2} [\sin(4y) - \sin(y)]\,dy$$
$$= \frac{1}{2}\left[-\frac{1}{4}\cos(4y) + \cos(y)\right]_0^{\pi/2} dy$$
$$= \frac{1}{2}\left[-\frac{1}{4}\cos(2\pi) + \cos\left(\frac{\pi}{2}\right) - \left(-\frac{1}{4}\cos(0) + \cos(0)\right)\right]$$
$$= \frac{1}{2}\left[-\frac{1}{4} + 0 + \frac{1}{4} - 1\right] = \boxed{-\frac{1}{2}}$$

Ex.4 Evaluate $\int_1^3 \int_{-1}^0 \frac{1}{1+x+y}\, dy\, dx$.

Start with u-sub: $u = (1+x) + y$ (constants), $du = dy$

$$\int_1^3 \int_{y=-1}^{y=0} \frac{1}{u}\, du\, dx$$
$$= \int_1^3 \left[\ln u\right]_{y=-1}^{y=0} dx$$
$$= \int_1^3 \left[\ln(1+x+y)\right]_{y=-1}^{y=0} dx$$
$$= \int_1^3 \left[\ln(1+x) - \ln(x)\right] dx$$

$\int \ln(1+x)\, dx$

By parts: $u = \ln(1+x)$, $du = \frac{1}{1+x}\, dx$ $\quad$ $dv = dx$, $v = x$

$$= x\ln(1+x) - \underbrace{\int \frac{x}{1+x}\, dx}_{\text{still messy!}}$$

Instead: $\int_1^3 \ln(1+x)\, dx - \int_1^3 \ln(x)\, dx$

$t = 1+x$
$dt = dx$
$x = 1: \ t = 2$
$x = 3: \ t = 4$

$$\int_2^4 \ln(t)\, dt - \int_1^3 \ln(x)\, dx$$

Now: $\int \ln(x)\, dx$ by parts: $u = \ln(x)$, $du = \frac{1}{x}\, dx$ $\quad$ $dv = dx$, $v = x$

$$= x\ln(x) - \int x \cdot \frac{1}{x}\, dx$$
$$= x\ln(x) - \int dx$$
$$= x\ln(x) - x$$

$$= \left[t\ln t - t\right]_{t=2}^{t=4} - \left[x\ln(x) - x\right]_{t=1}^{t=3}$$
$$= \left[4\ln 4 - 4 - (2\ln 2 - 2)\right] - \left[3\ln(3) - 3 - (\cancel{1\ln(1)}^{\,0} - 1)\right]$$
$$= 4\ln 4 - 2\ln 2 - \cancel{2} - 3\ln 3 + \cancel{2}$$
$$= \boxed{4\ln 4 - 2\ln 2 - 3\ln 3}$$

OR: $\ln\left(\frac{4^{\cancel{4}\,3}}{\cancel{2^2}}\right) - \ln(3^3) = \ln\left(\left(\frac{4}{3}\right)^3\right) = \boxed{\ln\left(\frac{64}{27}\right)}$

Def. The <u>average value</u> of f over a region R is

$$\boxed{\bar{f} = f_{ave} = f_{avg} = \frac{1}{\text{area of } R} \iint_R f(x,y)\, dA}$$

The average value can be interpreted as the height of a rectangular prism with base R whose volume is equivalent to $\iint_R f(x,y)\,dA$, i.e.

$$f_{ave} \cdot \text{area of } R = \iint_R f(x,y)\,dA$$

Ex.5 Find the average value of $f(x,y) = \sin x \sin y$ over $R = \{(x,y) : 0 \le x \le \pi,\ 0 \le y \le \pi\}$

Area of $R = (\pi - 0)(\pi - 0) = \pi^2$

$$\begin{aligned}\int_0^\pi \int_0^\pi \sin x \sin y \, dy\, dx &= \int_0^\pi \left[-\cos x \sin y\right]_{x=0}^{x=\pi} dy \\ &= \int_0^\pi [\sin y + \sin y]\, dy \\ &= \int_0^\pi 2\sin y \, dy \\ &= -2\cos y \Big|_0^\pi \\ &= 4\end{aligned}$$

$$\boxed{\bar{f} = \frac{4}{\pi^2}}$$

Ex.6 Find the average squared distance between the points of $R = \{(x,y) \mid 0 \le x \le 3,\ 0 \le y \le 3\}$ and the point $(3, 3)$.

$f(x,y) = $ distance squared $= (x-3)^2 + (y-3)^2$

Area of $R = 3 \cdot 3 = 9$

$$\int_0^3 \int_0^3 [(x-3)^2 + (y-3)^2]\, dy\, dx = \int_0^3 (3(x-3)^2 + 9)\, dx = 54$$

$$\boxed{\bar{f} = 6}$$

Ex.7 Evaluate the following integrals using symmetry arguments. Let $R = \{(x,y) \mid -a \le x \le a,\ -b \le y \le b\}$, $a, b > 0$.

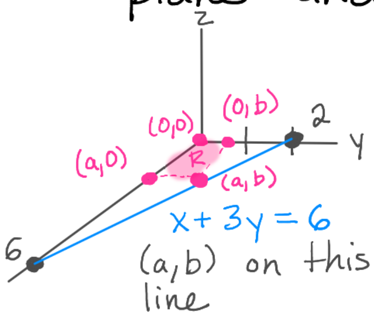
(a) $\displaystyle \iint_R xy e^{-(x^2+y^2)}\, dA = \boxed{0}$

If $f(x,y) = xye^{-(x^2+y^2)}$, then $f(-x,y) = -f(x,y)$, so the integrals in quadrants II and III cancel the integrals over quadrants I and IV.

(b) $\iint_R \underbrace{\frac{\sin(x-y)}{x^2+y^2+1}}_{=f(x,y)} \, dA$ Note: $f(-x,-y) = -f(x,y)$, so Q IV cancels Q I, and Q III cancels Q II.

$\underbrace{\qquad\qquad\qquad}_{=0}$

Ex.8 Consider the plane $x+3y+z=6$ over the rectangle with vertices $(0,0)$, $(a,0)$, $(0,b)$, and (a,b), where the vertex (a,b) lies on the line where the plane intersects the xy-plane (so $a+3b=6$). Find the point (a,b) for which the volume of the solid between the plane and R is a maximum.



height is $z = f(x,y) = 6 - x - 3y$

$$V(a,b) = \int_0^b \int_0^a (6-x-3y)\,dx\,dy$$
$$= \int_0^b \left(6x - \tfrac{1}{2}x^2 - 3xy\right)\Big|_0^a dy$$
$$= \int_0^b \left(6a - \tfrac{a^2}{2} - 3ay\right) dy$$
$$= 6ay - \tfrac{a^2}{2}y - \tfrac{3}{2}ay^2\Big|_0^b$$
$$= 6ab - \frac{a^2b}{2} - \frac{3ab^2}{2}$$

Maximize subject to $a+3b=6$:

$$V(6-3b,\, b) = 6(6-3b)b - \tfrac{1}{2}(6-3b)^2 b - \tfrac{3}{2}(6-3b)b^2$$
$$= 36b - 18b^2 - \tfrac{1}{2}(36b - 36b^2 + 9b^3) - \tfrac{3}{2}(6b^2 - 3b^3)$$
$$= 18b - 9b^2$$

$$\frac{dV}{db} = 18 - 18b = 0$$
$$\Rightarrow \quad b = 1 \quad \Rightarrow \quad a = 3 \qquad \boxed{(3,1)}$$