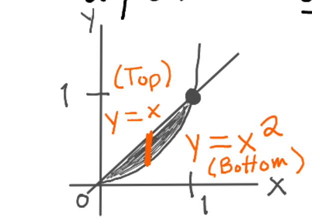
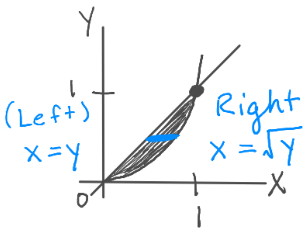
§16.2 Double Integrals over General Regions

$$\int_{\text{Left}\ x=a}^{\text{Right}\ x=b} \int_{\text{Bottom}\ y=g_1(x)}^{\text{Top}\ y=g_2(x)} f(x,y)\, dy\, dx \;=\; \int_{\text{Bottom}\ y=c}^{\text{Top}\ y=d} \int_{\text{Left}\ x=h_1(y)}^{\text{Right}\ x=h_2(y)} f(x,y)\, dx\, dy$$

<u>Ex.1</u> Write the double integral of $f(x,y)$ over the region R bounded by $y=x^2$ and $y=x$ with order of integration $dy\,dx$ <u>and</u> $dx\,dy$



$$\int_{\text{Left}}^{\text{Right}} \int_{\text{Bottom}}^{\text{Top}} f(x,y)\, dy\, dx = \boxed{\int_0^1 \int_{x^2}^{x} f(x,y)\, dy\, dx}$$



$$\int_{\text{Bottom}}^{\text{Top}} \int_{\text{Left}}^{\text{Right}} f(x,y)\, dx\, dy = \boxed{\int_0^1 \int_{y}^{\sqrt{y}} f(x,y)\, dx\, dy}$$

<u>Ex.2</u> Evaluate $\int_0^{\ln 2} \int_{e^y}^{2} \frac{y}{x}\, dx\, dy$.

Try as written: $\int_0^{\ln 2} \Big[y \ln(x) \Big]_{x=e^y}^{x=2} dy$

$$= \int_0^{\ln 2} \left[y \ln 2 - y \ln(e^y) \right] dy$$

$$= \int_0^{\ln 2} (y \ln 2 - y^2)\, dy$$

$$= \left[\tfrac{1}{2} y^2 \ln 2 - \tfrac{1}{3} y^3 \right]_0^{\ln 2}$$

$$= \tfrac{1}{2} (\ln 2)^3 - \tfrac{1}{3} (\ln 2)^3 = \boxed{\tfrac{1}{6} (\ln 2)^3}$$

<u>Ex.3</u> Evaluate $\int_{\pi/2}^{\pi} \int_0^{y^2} \cos\left(\frac{x}{y}\right) dx\, dy$.

Try as written: $u = \frac{x}{y}$, $du = \frac{1}{y}\, dx$

$$= \int_{\pi/2}^{\pi} \int_0^{y^2} y \cos u\, du\, dy$$

$$= \int_{\pi/2}^{\pi} \left[y \sin\left(\tfrac{x}{y}\right) \right]_{x=0}^{x=y^2} dy$$

$$= \int_{\pi/2}^{\pi} y \sin y\, dy$$

$u = y$, $dv = \sin y\, dy$
$du = dy$, $v = -\cos y$

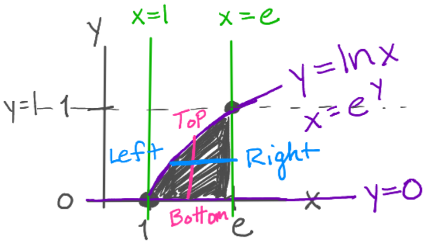
$$= -y \cos y \Big|_{\pi/2}^{\pi} + \int_{\pi/2}^{\pi} \cos y\, dy$$

$$= -y\cos y + \sin y \Big|_{\pi/2}^{\pi} = \boxed{\pi - 1}$$

Ex.4 Evaluate $\iint_R x^3\,dA$ where $R = \{(x,y) : \underbrace{1 \le x \le e}_{\text{constants}},\ 0 \le y \le \ln x\}$

$$\int_1^e \int_0^{\ln x} x^3\,dy\,dx = \int_1^e \left[x^3 y\right]_0^{\ln x} dx$$

$$= \int_1^e x^3 \ln x\,dx \rightarrow \text{Integration by parts! Try switching the order.}$$



$$\int_{y=0}^{y=1} \int_{x=e^y}^{x=e} x^3\,dx\,dy = \int_0^1 \left[\tfrac{1}{4}x^4\right]_{e^y=x}^{e=x} dy$$

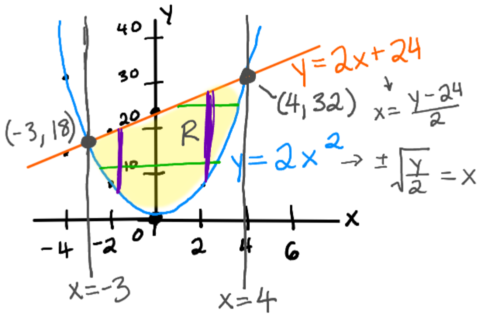
$$= \tfrac{1}{4}\int_0^1 (e^4 - e^{4y})\,dy \rightarrow \text{u-sub!}$$

$$= \tfrac{1}{4}\left[e^4 y - \tfrac{1}{4}e^{4y}\right]_0^1$$

$$= \tfrac{1}{4}\left(e^4 - \tfrac{1}{4}e^4 - (0 - \tfrac{1}{4})\right)$$

$$= \tfrac{1}{4}e^4 - \tfrac{1}{16}e^4 + \tfrac{1}{16} = \boxed{\frac{3e^4+1}{16}}$$

Ex.5 Calculate $\iint_R xy\,dA$ where R is the region below.



Find where intersect: $y = 2x+24$ and $y = 2x^2$

$$2x + 24 = 2x^2$$
$$0 = x^2 - x - 12$$
$$0 = \underbrace{(x-4)}_{x=4}\underbrace{(x+3)}_{x=-3}$$

Notice that horizontal lines don't have the same functions on left and right, so we would have to separate the integrals.

$$\int_0^{18}\int_{-\sqrt{\frac{y}{2}}}^{\sqrt{\frac{y}{16}}} xy\,dx\,dy + \int_{18}^{32}\int_{\frac{y-24}{2}}^{\sqrt{\frac{y}{2}}} xy\,dx\,dy$$

The vertical lines always have Top: $y = 2x+24$ and Bottom: $y=2x^2$.

$$\int_{-3}^{4}\int_{2x^2}^{2x+24} xy\,dy\,dx = \int_{-3}^{4}\left[\tfrac{1}{2}xy^2\right]_{y=2x^2}^{y=2x+24} dx$$

$$= \int_{-3}^{4} (2x^3 + 48x^2 + 288x - 2x^5)\,dx$$

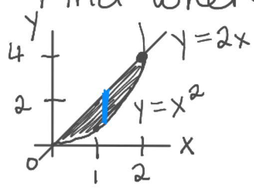
$$= \boxed{\frac{8575}{6}}$$

Ex.6 Find the volume of the solid that lies under $z = x^2 + y^2$ and above the region in the xy-plane bounded by $y = 2x$ and $y = x^2$.

Volume $= \iint_R \text{height}\, dA$; height $= x^2 + y^2$.

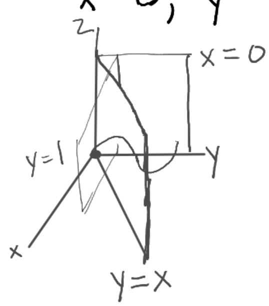
Find where $y = 2x$ and $y = x^2$ intersect:

$$2x = x^2$$
$$0 = \underbrace{x}_{x=0}\underbrace{(x-2)}_{x=2}$$



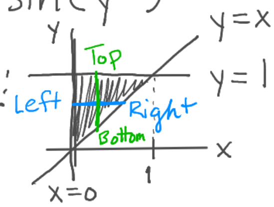
$$V = \int_0^2 \int_{x^2}^{2x} (x^2 + y^2)\, dy\, dx$$
$$= \int_0^2 \left[x^2 y + \tfrac{1}{3}y^3\right]_{y=x^2}^{y=2x} dx$$
$$= \int_0^2 \left(\tfrac{14}{3}x^3 - x^4 - \tfrac{1}{3}x^6\right) dx = \boxed{\tfrac{216}{35}}$$

Ex.7 Find the volume under the surface $z = \sin(y^2)$ and the planes $x = 0$, $y = 0$, $y = 1$, and $y = x$.



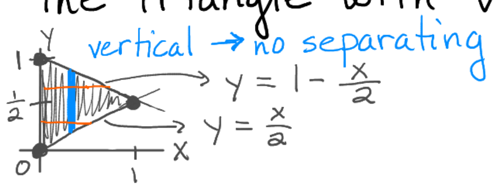
height $= \sin(y^2)$

xy-plane:



$$V = \int_0^1 \int_{x=0}^{x=y} \sin(y^2)\, dx\, dy \quad \text{or} \quad V = \int_0^1 \underbrace{\int_{y=x}^{y=1} \sin(y^2)\, dy}_{\text{can't integrate easily}}\, dx$$
$$= \int_0^1 \left[x \sin(y^2)\right]_{x=0}^{x=y} dy$$
$$= \int_0^1 y \sin(y^2)\, dy$$
$$= -\tfrac{1}{2}\cos(y^2)\Big|_0^1 = \boxed{\tfrac{1}{2}(1 - \cos(1))} \approx 0.23$$

Ex.8 Find the volume under the plane $x + 2y + z = 2$ and above the triangle with vertices $(0,0)$, $(0,1)$, and $(1, \frac{1}{2})$.



horizontal → separating

$$V = \int_0^1 \int_{x/2}^{1-x/2} (2 - x - 2y)\, dy\, dx$$
$$= \int_0^1 \left[2y - xy - y^2\right]_{y=x/2}^{y=1-x/2} dx$$
$$= \int_0^1 (1 - 2x + x^2)\, dx$$
$$= \left(x - x^2 + \tfrac{1}{3}x^3\right)\Big|_0^1 = \boxed{\tfrac{1}{3}}$$