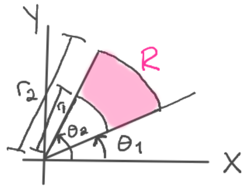
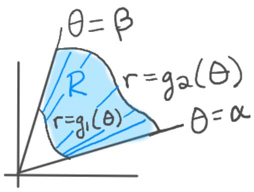
§16.3 Double Integrals in Polar Coordinates



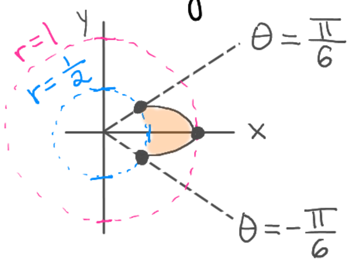
$$\iint_R f(x,y)\,\underbrace{dA}_{\substack{\text{Cartesian}\\ \text{coordinates}\\ = dx\,dy \text{ or}\\ dy\,dx}} = \int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2} f(r\cos\theta, r\sin\theta)\,\underbrace{r\,dr\,d\theta}_{\substack{dA\\ \text{in polar}\\ \text{coordinates}}}$$



$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

Translate (x,y) into (r,θ) using $x = r\cos\theta$, $y = r\sin\theta$ $\quad r^2 = x^2 + y^2$, $\frac{y}{x} = \tan\theta$

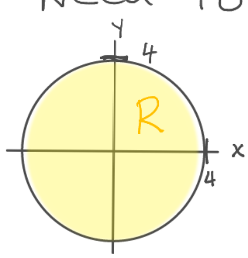
Ex.1 Sketch in the xy-plane the region of integration for the integral $\int_{-\pi/6}^{\pi/6}\int_{1/2}^{\cos(2\theta)} g(r,\theta)\,r\,dr\,d\theta$.



θ	$\cos(2\theta)$
$-\frac{\pi}{6}$	$\cos(-\frac{\pi}{3}) = \frac{1}{2}$
0	1
$\frac{\pi}{6}$	$\cos(\frac{\pi}{3}) = \frac{1}{2}$

Ex.2 Find the volume of the solid bounded by the surface $f(x,y) = e^{-(x^2+y^2)/8} - e^{-2}$. $\rightarrow$ Can't integrate $\int e^{-(x^2+y^2)}\,dx$ or $\int e^{-(x^2+y^2)}\,dy$

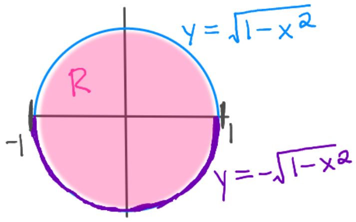
Need to figure out the region in the xy-plane.



$$0 = e^{-(x^2+y^2)/8} - e^{-2}$$
$$e^{-2} = e^{-(x^2+y^2)/8}$$
$$2 = \tfrac{1}{8}(x^2+y^2)$$
$$16 = x^2 + y^2$$

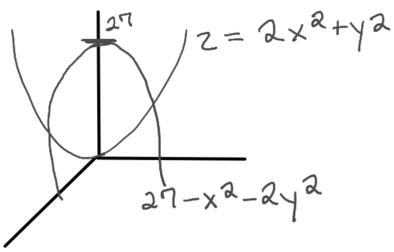
$$V = \iint_R f(x,y)\,dA = \int_0^{2\pi}\int_0^4 (e^{-r^2/8} - e^{-2})\,r\,dr\,d\theta$$
$$= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^4 (re^{-r^2/8} - re^{-2})\,dr\right)$$
$$= 2\pi\left(-4e^{-r^2/8} - \tfrac{e^{-2}}{2}r^2\right)\Big|_0^4$$
$$= 2\pi\left(-4e^{-2} - 8e^{-2} - (-4 - 0)\right)$$
$$= \boxed{8\pi(1 - 3e^{-2})}$$

Ex.3 Evaluate $\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (x^2+y^2)^{3/2}\,dy\,dx$ using polar coordinates.

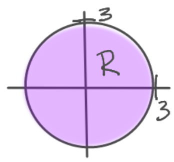


$$\int_0^{2\pi}\int_0^1 (r^2)^{3/2}\, r\,dr\,d\theta$$
$$=\left(\int_0^{2\pi} d\theta\right)\left(\int_0^1 r^4\,dr\right)$$
$$=\boxed{\frac{2\pi}{5}}$$

Ex.4 Find the volume of the solid bounded by the paraboloids $z=2x^2+y^2$ and $z=27-x^2-2y^2$.



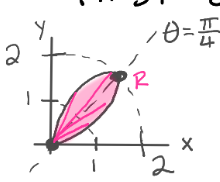
Find intersection: $2x^2+y^2=27-x^2-2y^2$
$x^2+y^2=9$



Height $=27-x^2-2y^2-(2x^2+y^2)=27-3x^2-3y^2=27-3r^2$

$$\iint_R f(x,y)\,dA=\int_0^{2\pi}\int_0^3 (27-3r^2)\,r\,dr\,d\theta$$
$$=2\pi\left[\frac{27}{2}r^2-\frac{3}{4}r^4\right]_0^3$$
$$=\boxed{\frac{243\pi}{2}}$$

Ex.5 Sketch the region and write $\iint_R g(r,\theta)\,dA$ where R is the region inside the leaf of the rose $r=2\sin 2\theta$ in the first quadrant.

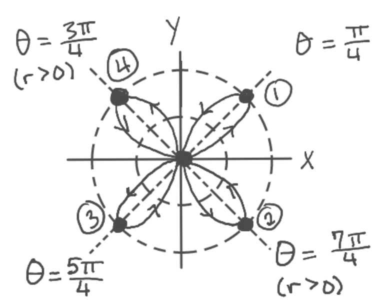


θ	r
0	0
$\frac{\pi}{4}$	2
$\frac{\pi}{2}$	0

$$\int_0^{\pi/2}\int_0^{2\sin 2\theta} g(r,\theta)\,r\,dr\,d\theta$$

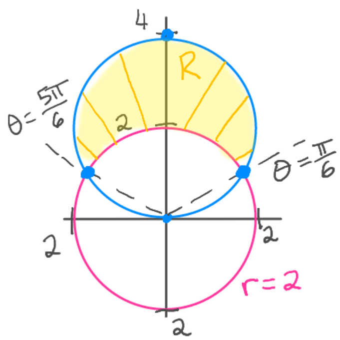
How many leaves?

θ	$2\sin 2\theta$
$\frac{3\pi}{4}$	-2
π	0
$\frac{5\pi}{4}$	2
$\frac{3\pi}{2}$	0
$\frac{7\pi}{4}$	-2



4 leaves

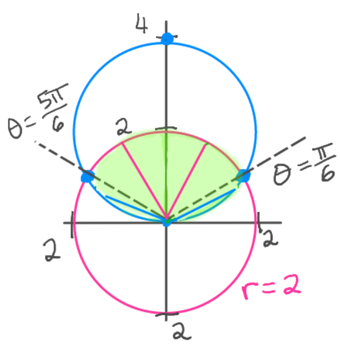
Ex.6 Sketch the region outside the circle $r=2$ and inside the circle $r=4\sin\theta$ and set up the iterated integral $\iint_R g(r,\theta)\,dA$ in polar coordinates.



θ	$4\sin\theta$
0	0
$\frac{\pi}{6}$	2
$\frac{\pi}{2}$	4
$\frac{5\pi}{6}$	2
π	0
$\frac{7\pi}{6}$	-2
$\frac{3\pi}{2}$	-4
$\frac{11\pi}{6}$	-2

$$2=4\sin\theta$$
$$\tfrac{1}{2}=\sin\theta$$
$$\theta=\tfrac{\pi}{6},\ \tfrac{5\pi}{6}$$

$$\boxed{\int_{\pi/6}^{5\pi/6}\int_{2}^{4\sin\theta} g(r,\theta)\,r\,dr\,d\theta}$$



Sketch the region inside both $r=2$ and $r=4\sin\theta$ and set up $\iint_R g(r,\theta)\,dA$.

The rays touch different curves, so we can write $\iint g(r,\theta)\,dA + \iint g(r,\theta)\,dA + \iint g(r,\theta)\,dA$

$$=\int_0^{\pi/6}\int_0^{4\sin\theta} g(r,\theta)\,r\,dr\,d\theta+\int_{\pi/6}^{\frac{5\pi}{6}}\int_0^{2} g(r,\theta)\,r\,dr\,d\theta+\int_{\frac{5\pi}{6}}^{\pi}\int_0^{4\sin\theta} g(r,\theta)\,r\,dr\,d\theta$$

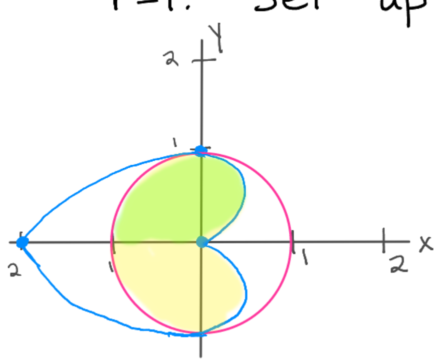
or we can write $\iint g(r,\theta)\,dA - \iint g(r,\theta)\,dA$

$$=\int_0^{\pi}\int_0^{4\sin\theta} g(r,\theta)\,r\,dr\,d\theta-\int_{\pi/6}^{5\pi/6}\int_2^{4\sin\theta} g(r,\theta)\,r\,dr\,d\theta$$

To find the <u>area</u> of the region, we can use symmetry:

$$A=2\left[\int_0^{\pi/2}\int_0^{4\sin\theta} r\,dr\,d\theta-\int_{\pi/6}^{\pi/2}\int_2^{4\sin\theta} r\,dr\,d\theta\right]$$

Ex.7 Find the area of region inside both $r=1-\cos\theta$ and $r=1$. Set up as efficiently as possible.



θ	$1-\cos\theta$
0	0
$\frac{\pi}{2}$	1
π	2
$\frac{3\pi}{2}$	1

Total Area $= 2 \cdot$ Area above x-axis

$$= \int_0^{\pi/2}\int_0^{1-\cos\theta} r\,dr\,d\theta + \int_{\pi/2}^{\pi}\int_0^1 r\,dr\,d\theta$$

$$= \int_0^{\pi/2} \tfrac{1}{2}(1-\cos\theta)^2\,d\theta + \frac{\pi}{4}$$

$$= \frac{\pi}{4} + \frac{1}{2}\int_0^{\pi/2}(1-2\cos\theta+\cos^2\theta)\,d\theta \qquad \text{Use } \cos^2\theta = \tfrac{1}{2}(1+\cos 2\theta)$$

$$= \frac{\pi}{4} + \frac{1}{2}\left(\frac{3}{2}\theta - 2\sin\theta + \frac{1}{4}\sin(2\theta)\right)\Big|_0^{\pi/2}$$

$$= \frac{\pi}{4} + \frac{1}{2}\left(\frac{3\pi}{4} - 2 + 0 - (0-0+0)\right)$$

$$= \frac{\pi}{4} + \frac{3\pi}{8} - 1 = \frac{5\pi}{8} - 1$$

$$\boxed{\text{Total Area} = \frac{5\pi}{4} - 2}$$

Trig Identities:

$$\sin^2 x + \cos^2 x = 1$$

divide by $\cos^2 x$ | divide by $\sin^2 x$

$$\tan^2 x + 1 = \sec^2 x$$

$$1 + \cot^2 x = \csc^2 x$$

$$\sin(2x) = 2\sin x \cos x$$

$$\cos(2x) = \cos^2 x - \sin^2 x$$

$$\cos^2 x = \tfrac{1}{2}(1+\cos(2x))$$

$$\sin^2 x = \tfrac{1}{2}(1-\cos(2x))$$