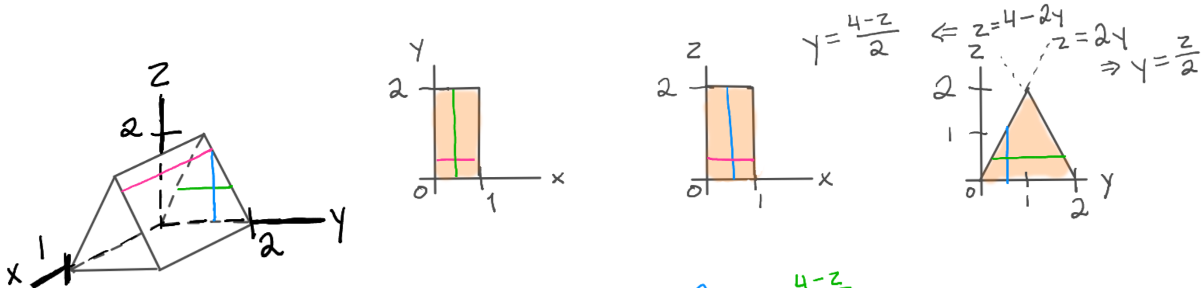
§16.4 Triple Integrals

$\iiint_D f(x,y,z)\,dV$ can be written 6 ways with $dV = dx\,dy\,dz,\ dy\,dx\,dz,\ dz\,dx\,dy,\ dx\,dz\,dy;\ dy\,dz\,dx,\ dz\,dy\,dx.$

Ex.1 Write six iterated integrals $\iiint_D f(x,y,z)\,dV$ where D is the triangular prism bounded by $z=0$, $z=2y$, $z=-2y+4$, $x=0$ and $x=1$.



$$\int_{\substack{\text{Bottom}\\(\#)}}^{\text{Top}} \int_{\substack{\text{Left}\\(z)}}^{\text{Right}} \int_{\substack{\text{Back}\\(y,z)}}^{\text{Front}} f(x,y,z)\,dx\,dy\,dz = \int_{z=0}^{z=2} \int_{y=z/2}^{y=\frac{4-z}{2}} \int_{x=0}^{x=1} f(x,y,z)\,dx\,dy\,dz$$

$$\int_{\substack{\text{Left}\\(\#)}}^{\text{Right}} \int_{\substack{\text{Bottom}\\(y)}}^{\text{Top}} \int_{\substack{\text{Back}\\(y,z)}}^{\text{Front}} f(x,y,z)\,dx\,dz\,dy = \int_{y=0}^{y=1} \int_{z=0}^{z=2y} \int_{x=0}^{x=1} f(x,y,z)\,dx\,dz\,dy + \int_{y=1}^{y=2} \int_{z=0}^{z=4-2y} \int_{x=0}^{x=1} f(x,y,z)\,dx\,dz\,dy$$

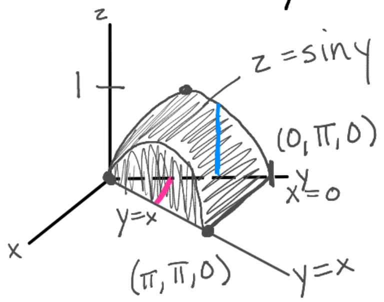
$$\int_{\substack{\text{Bottom}\\(\#)}}^{\text{Top}} \int_{\substack{\text{Back}\\(y)}}^{\text{Front}} \int_{\substack{\text{Left}\\(x,z)}}^{\text{Right}} f(x,y,z)\,dy\,dx\,dz = \int_{z=0}^{z=2} \int_{x=0}^{x=1} \int_{y=z/2}^{y=\frac{4-z}{2}} f(x,y,z)\,dy\,dx\,dz$$

$$\int_{\substack{\text{Back}\\(\#)}}^{\text{Front}} \int_{\substack{\text{Bottom}\\(x)}}^{\text{Top}} \int_{\substack{\text{Left}\\(x,z)}}^{\text{Right}} f(x,y,z)\,dy\,dz\,dx = \int_{x=0}^{x=1} \int_{z=0}^{z=2} \int_{y=z/2}^{y=\frac{4-z}{2}} f(x,y,z)\,dy\,dz\,dx$$

$$\int_{\substack{\text{Left}\\(\#)}}^{\text{Right}} \int_{\substack{\text{Back}\\(y)}}^{\text{Front}} \int_{\substack{\text{Bottom}\\(x,y)}}^{\text{Top}} f(x,y,z)\,dz\,dx\,dy = \int_{y=0}^{y=1} \int_{x=0}^{x=1} \int_{z=0}^{z=2y} f(x,y,z)\,dz\,dx\,dy + \int_{y=1}^{y=2} \int_{x=0}^{x=1} \int_{z=0}^{z=4-2y} f(x,y,z)\,dz\,dx\,dy$$

$$\int_{\substack{\text{Back}\\(\#)}}^{\text{Front}} \int_{\substack{\text{Left}\\(x)}}^{\text{Right}} \int_{\substack{\text{Bottom}\\(x,y)}}^{\text{Top}} f(x,y,z)\,dz\,dy\,dx = \int_{x=0}^{x=1} \int_{y=0}^{y=1} \int_{z=0}^{z=2y} f(x,y,z)\,dz\,dy\,dx + \int_{x=0}^{x=1} \int_{y=1}^{y=2} \int_{z=0}^{z=4-2y} f(x,y,z)\,dz\,dy\,dx$$

Ex. 2 Use a triple integral to find the volume of the solid in the first octant formed when the cylinder $z = \sin y$ for $0 \le y \le \pi$ is sliced by the planes $y = x$ and $x = 0$.



$$V = \iiint dV$$

$$= \int_{y=0}^{y=\pi} \int_{x=0}^{x=y} \int_{z=0}^{z=\sin y} dz\, dx\, dy$$

$$= \int_0^{\pi} \int_0^{y} \left[z \right]_{z=0}^{z=\sin y} dx\, dy$$

$$= \int_0^{\pi} \int_0^{y} \sin y \; dx\, dy$$

$$= \int_0^{\pi} \left[x \sin y \right]_{x=0}^{x=y} dy$$

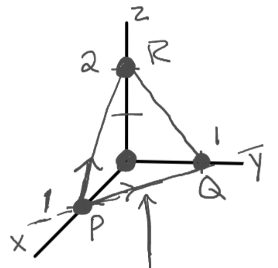
$u = y \qquad dv = \sin y\, dy$

$du = dy \qquad v = -\cos y$

$$= \int_0^{\pi} y \sin y \; dy$$

$$= -y\cos y + \sin y \Big|_0^{\pi} = \boxed{\pi}$$

Ex. 3 Find the volume of the tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, and $(0,0,2)$ using a triple integral



Line in xy-plane
$y = 1 - x$

Find the equation of the plane through $P(1,0,0)$, $Q(0,1,0)$ and $R(0,0,2)$.

Plane contains vectors $\overrightarrow{PQ} = \langle -1, 1, 0 \rangle$ and $\overrightarrow{PR} = \langle -1, 0, 2 \rangle$.

A vector normal to the plane is

$$\vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & 0 \\ -1 & 0 & 2 \end{vmatrix} = \langle 2, 2, 1 \rangle$$

Equation of plane: $2(x-1) + 2(y-0) + 1(z-0) = 0$

$$2x - 2 + 2y + z = 0$$

$$V = \int_0^1 \int_0^{1-x} \int_0^{2-2x-2y} dz\, dy\, dx$$

$$= \int_0^1 \int_0^{1-x} (2 - 2x - 2y)\, dy\, dx$$

$$= \int_0^1 \left[2y - 2xy - y^2 \right]_{y=0}^{y=1-x} dx$$

$$= \int_0^1 (1 - 2x + x^2)\, dx$$

$$= x - x^2 + \tfrac{1}{3} x^3 \Big|_0^1$$

$$= 1 - 1 + \tfrac{1}{3}$$

$$= \boxed{\tfrac{1}{3}}$$

Ex.4 Evaluate $\iiint_D xe^{-y}\,dV$ where $D = \{(x,y,z) : 0 \le x \le 2z,\ 0 \le y \le \ln x,\ 1 \le z \le 2\}$.

$$\int_1^2 \int_0^{2z} \int_0^{\ln x} xe^{-y}\,dy\,dx\,dz = \int_1^2 \int_0^{2z} \left[-xe^{-y}\right]_{y=0}^{y=\ln x} dx\,dz$$

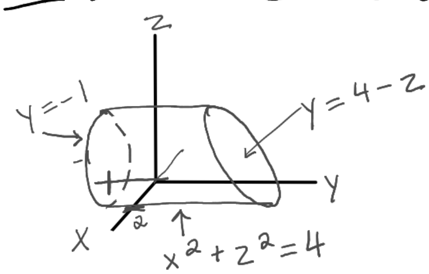
$$= \int_1^2 \int_0^{2z} \left[-xe^{-\ln x} + x\right] dx\,dz \qquad e^{-\ln x} = e^{\ln(1/x)} = \frac{1}{x}$$

$$= \int_1^2 \int_0^{2z} (x-1)\,dx\,dz$$

$$= \int_1^2 \left[\tfrac{1}{2}x^2 - x\right]_{x=0}^{x=2z} dz$$

$$= \int_1^2 (2z^2 - 2z)\,dz = \boxed{\tfrac{14}{3} - 3}$$

Ex.5 Evaluate $\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{-1}^{4-z} dy\,dz\,dx$



$$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (5-z)\,dz\,dx$$

Using polar coordinates in the xz-plane:

$z = r\sin\theta$

$x = r\cos\theta$

$dz\,dx = r\,dr\,d\theta$

$$\int_0^{2\pi} \int_0^2 (5 - r\sin\theta)\,r\,dr\,d\theta = \int_0^{2\pi} \left[\tfrac{5}{2}r^2 - \tfrac{1}{3}r^3\sin\theta\right]_{r=0}^{r=2} d\theta$$

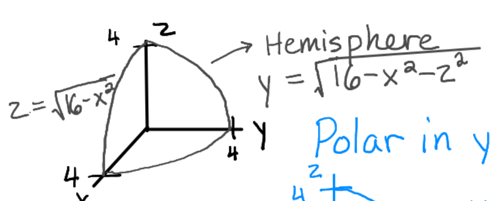
$$= \int_0^{2\pi} \left(10 - \tfrac{8}{3}\sin\theta\right) d\theta$$

$$= 10\theta + \tfrac{8}{3}\cos\theta \Big|_0^{2\pi}$$

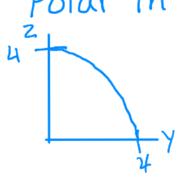
$$= 20\pi + \tfrac{8}{3} - \left(0 + \tfrac{8}{3}\right)$$

$$= \boxed{20\pi}$$

Ex.6 Rewrite $\int_0^4 \int_0^{\sqrt{16-x^2}} \int_0^{\sqrt{16-x^2-z^2}} dy\,dz\,dx$ in the order $dx\,dy\,dz$ and evaluate.



Polar in yz-plane



$y = 4\cos\theta$

$z = 4\sin\theta$

$$= \int_{z=0}^{z=4} \int_{y=0}^{y=\sqrt{16-z^2}} \int_{x=0}^{x=\sqrt{16-y^2-z^2}} dx\,dy\,dz$$

$$= \int_0^4 \int_0^{\sqrt{16-z^2}} \sqrt{16-y^2-z^2}\,dy\,dz$$

$$= \int_0^{\pi/2} \int_0^4 \sqrt{16-r^2}\,r\,dr\,d\theta$$

$$= \frac{\pi}{2}\left[-\frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{3}(16-r^2)^{3/2}\right]_0^4 = \boxed{\frac{32\pi}{3}}$$

Note: In calc 2, we found the area between curves $f_1(x)$ and $f_2(x)$ with $f_2(x) \geq f_1(x)$ on $[a,b]$ by computing

$$A = \int_a^b \left(f_2(x) - f_1(x)\right) dx \quad \text{(or in terms of } y\text{)}$$

Now, we compute the double integral

$$\begin{aligned} A &= \int_a^b \int_{f_1(x)}^{f_2(x)} dy\,dx \qquad \text{(or with } dx\,dy\text{)} \\ &= \int_a^b \left[y\right]_{y=f_1(x)}^{y=f_2(x)} dx \\ &= \int_a^b \left(f_2(x) - f_1(x)\right) dx \end{aligned}$$

In §16.1, 16.2, and 16.3, we found the volume above the surface $z = f_1(x,y)$ (or the xy-plane $z=0$) and below the surface $z = f_2(x,y)$ by computing the double integral

$$V = \iint_R \left(f_2(x,y) - f_1(x,y)\right) dA$$

In this section, we use a triple integral:

$$\begin{aligned} V &= \iint_R \int_{f_1(x,y)}^{f_2(x,y)} dz\,dA \\ &= \iint_R \left[z\right]_{z=f_1(x,y)}^{z=f_2(x,y)} dA \\ &= \iint_R \left(f_2(x,y) - f_1(x,y)\right) dA \end{aligned}$$

(Another)
Note: We can use symmetry to compute volumes or areas, but when evaluating a function <u>over</u> a symmetric region or surface, use caution.

Recall (from calc 1), this works differently for even and odd functions:

$$\begin{aligned} \int_{-1}^1 x\,dx &= \tfrac{1}{2}x^2\Big|_{-1}^1 \\ &= \tfrac{1}{2}\left(1^2 - (-1)^2\right) \\ &= 0 \end{aligned}$$

$$\begin{aligned} \int_{-1}^1 x^2\,dx &= \tfrac{1}{3}x^3\Big|_{-1}^1 \\ &= \tfrac{1}{3}\left(1^3 - (-1)^3\right) \\ &= \tfrac{2}{3} \end{aligned}$$