§16.5 Triple Integrals in Cylindrical and Spherical Coordinates

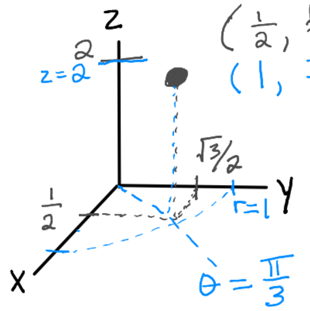
Cylindrical Coordinates: $(x, y, z) \longrightarrow (r, \theta, z)$

$x = r\cos\theta$ $\qquad$ $r^2 = x^2 + y^2$

$y = r\sin\theta$ $\qquad$ $\tan\theta = \frac{y}{x}$

$z = z$ $\qquad$ $z = z$

$$\boxed{\begin{array}{l} 0 \le \theta \le 2\pi \\ 0 \le r < \infty \\ -\infty < z < \infty \end{array}}$$



$(\frac{1}{2}, \frac{\sqrt{3}}{2}, 2)$ in rectangular coordinates

$(1, \frac{\pi}{3}, 2)$ in cylindrical coordinates

$$\boxed{\iiint_D f(x,y,z)\, dA = \iiint_D f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta}$$

Ex.1 Convert $(-1,-1,4)$ from rectangular to cylindrical coordinates

$x = -1$ $\qquad$ $x^2 + y^2 = r^2$ $\qquad$ $\tan\theta = \frac{y}{x}$

$y = -1$ $\qquad$ $1 + 1 = r^2$ $\qquad$ $\tan\theta = 1$

$z = 4$ $\qquad$ $\sqrt{2} = r$ $\qquad$ $\theta = \frac{5\pi}{4}$ (since $x, y < 0$)

$$\boxed{(\sqrt{2}, \tfrac{5\pi}{4}, 4)}$$

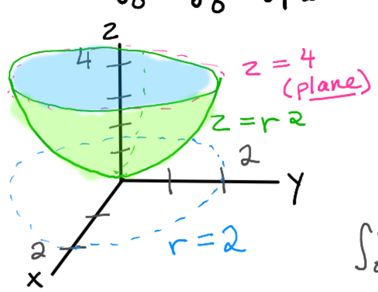
Ex.2 Convert $z^2 + 2xy - x^2 = y^2$ to cylindrical coordinates.

$$z^2 + 2r\cos\theta \cdot r\sin\theta - r^2\cos^2\theta = r^2\sin^2\theta$$

$$z^2 + 2r^2\cos\theta\sin\theta = r^2(\sin^2\theta + \cos^2\theta)$$

$$\boxed{z^2 + r^2\sin 2\theta = r^2}$$

Ex.3 Sketch the solid whose volume is given by the integral $\int_0^{2\pi}\int_0^2\int_{r^2}^4 r\, dz\, dr\, d\theta$ and evaluate.



$0 \le \theta \le 2\pi$ $\qquad$ $0 \le r \le 2$ $\qquad$ $r^2 \le z \le 4$

$0 \le r^2 \le 4$

$0 \le x^2 + y^2 \le 4$

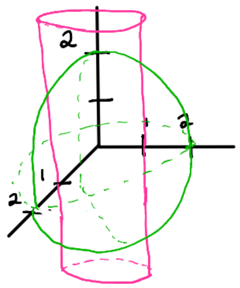
$$\int_0^{2\pi}\int_0^2 \Big[rz\Big]_{z=r^2}^{z=4} dr\, d\theta = \int_0^{2\pi}\int_0^2 (4r - r^3)\, dr\, d\theta$$

$$= \int_0^{2\pi} \Big[2r^2 - \tfrac{1}{4}r^4\Big]_{r=0}^{r=2} d\theta$$

$$= \int_0^{2\pi} (8-4)\, d\theta$$

$$= 4\theta \Big|_{\theta=0}^{\theta=2\pi} = \boxed{8\pi}$$

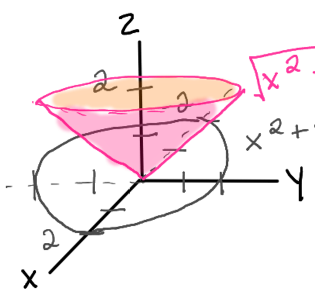
Ex.4 Find the volume of the solid that lies inside both $\underbrace{x^2+y^2=1}_{\text{cylinder}}$ and $\underbrace{x^2+y^2+z^2=4}_{\text{sphere}}$.



r and θ are in the xy-plane, so the circle of radius 1 is $0 \le r \le 1$ and $0 \le \theta \le 2\pi$.
For z, $z^2 = 4-(x^2+y^2) = 4-r^2$.
We can use $-\sqrt{4-r^2} \le z \le \sqrt{4-r^2}$ or because of symmetry we can double the volume for $0 \le z \le \sqrt{4-r^2}$.

$$V = 2\int_0^{2\pi}\int_0^1\int_0^{\sqrt{4-r^2}} r\,dz\,dr\,d\theta$$
$$= 2\int_0^{2\pi}\int_0^1 \left[rz\right]_{z=0}^{z=\sqrt{4-r^2}} dr\,d\theta$$
$$= 2\int_0^{2\pi}\int_0^1 \left(r\sqrt{4-r^2}\right) dr\,d\theta$$
$$= 4\pi\left[-\frac{1}{2}\cdot\frac{2}{3}(4-r^2)^{3/2}\right]_{r=0}^{r=1}$$
$$= 4\pi\left(-\frac{1}{3}\left(3^{3/2}-4^{3/2}\right)\right)$$
$$= \boxed{\frac{4\pi}{3}\left(8-3\sqrt{3}\right)}$$

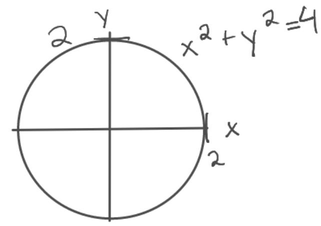
Ex.5 Evaluate using cylindrical coordinates:
$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2} (x^2+y^2)\,dz\,dy\,dx$$



$\sqrt{x^2+y^2} \le z \le 2$
$r \le z \le 2$

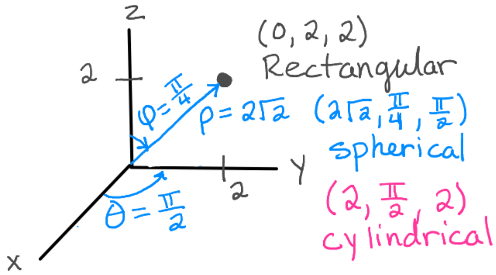
xy-plane:

$0 \le r \le 2$
$0 \le \theta \le 2\pi$



$$\int_0^{2\pi}\int_0^2\int_r^2 r^2\cdot r\,dz\,dr\,d\theta = 2\pi\int_0^2\left[r^3z\right]_{z=r}^{z=2} dr$$
$$= 2\pi\int_0^2 (2r^3 - r^4)\,dr$$
$$= 2\pi\left[\frac{1}{2}r^4 - \frac{1}{5}r^5\right]_{r=0}^{r=2}$$
$$= 2\pi\left(8 - \frac{32}{5}\right)$$
$$= \boxed{\frac{16\pi}{5}}$$

Spherical Coordinates: $(x, y, z) \longrightarrow (\rho, \varphi, \theta)$



$x = \rho \sin\varphi \cos\theta$

$y = \rho \sin\varphi \sin\theta$

$z = \rho \cos\varphi$

$\rho^2 = x^2 + y^2 + z^2$

Trig to get φ, θ

$0 \le \rho < \infty$
$0 \le \varphi \le \pi$
$0 \le \theta \le 2\pi$

ρ = distance from origin

φ = angle from positive z-axis

θ = angle from positive x-axis

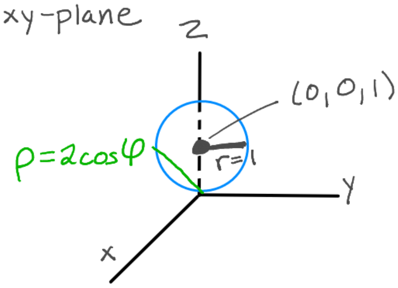
$$\boxed{\iiint_D f(x,y,z)\,dV = \iiint_D f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\,\rho^2\sin\phi\,d\rho\,d\varphi\,d\theta}$$

Ex.6 Sketch the set in spherical coordinates:

$$\{(\rho, \varphi, \theta) : \rho = 2\cos\varphi,\ \underbrace{0 \le \varphi \le \tfrac{\pi}{2}}_{\text{Above xy-plane}},\ \underbrace{0 \le \theta \le 2\pi}_{\text{Entire xy-plane}}\}.$$

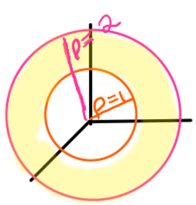
$$\rho^2 = \rho(2\cos\varphi)$$
$$\rho^2 = 2\rho\cos\varphi$$
$$x^2 + y^2 + z^2 = 2z$$
$$x^2 + y^2 + (z^2 - 2z + 1) = 1$$
$$x^2 + y^2 + (z-1)^2 = 1 \longrightarrow$$

A unit sphere centered at (0,0,1) in rectangular coord.



Ex.7 Evaluate in spherical coordinates: $\iiint_D \frac{dV}{(x^2+y^2+z^2)^{3/2}}$

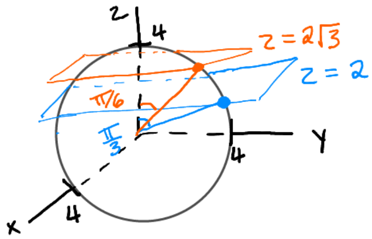
where D is the solid between the spheres of radius 1 and 2 centered at the origin.



$1 \le \rho \le 2$
$0 \le \varphi \le \pi$
$0 \le \theta \le 2\pi$

$$\int_0^{2\pi}\int_0^{\pi}\int_1^2 \frac{1}{(\rho^2)^{3/2}} \cdot \rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta$$
$$= 2\pi \int_0^{\pi}\int_1^2 \frac{1}{\rho} \cdot \sin\varphi \, d\rho\, d\varphi$$
$$= 2\pi \int_0^{\pi} [\sin\varphi \cdot \ln\rho]_{\rho=1}^{\rho=2} \, d\varphi$$
$$= 2\pi \cdot \ln 2 \int_0^{\pi} \sin\varphi \, d\varphi$$
$$= 2\pi \cdot \ln 2 \, [-\cos\varphi]_0^{\pi}$$
$$= \boxed{4\pi \cdot \ln 2}$$

Ex. 8 Find the volume of the part of the ball $\rho \le 4$ that lies between the planes $z = 2$ and $z = 2\sqrt{3}$.



Spherical coordinates:

Find φ for $z = 2\sqrt{3}$ and $z = 2$ to get the intersections of the planes with the sphere.

$$\begin{aligned} z &= 2\sqrt{3} \\ 4\cos\varphi &= 2\sqrt{3} \\ \cos\varphi &= \tfrac{\sqrt{3}}{2} \\ \varphi &= \tfrac{\pi}{6} \end{aligned} \qquad \begin{aligned} z &= 2 \\ 4\cos\varphi &= 2 \\ \cos\varphi &= \tfrac{1}{2} \\ \varphi &= \tfrac{\pi}{3} \end{aligned}$$

To find the volume <u>between</u> the planes, find the volume above $z = 2$ and subtract the volume above $z = 2\sqrt{3}$.

Above $z = 2$: ρ is between $z = 2 \Rightarrow \rho\cos\varphi = 2 \Rightarrow \rho = 2\sec\varphi$ and the sphere $\rho = 4$.

$$V_2 = \int_0^{2\pi}\int_0^{\pi/3}\int_{2\sec\varphi}^{4} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

$$= 2\pi \int_0^{\pi/3} \left[\tfrac{1}{3}\rho^3 \sin\varphi\right]_{\rho=2\sec\varphi}^{\rho=4} d\varphi$$

$$= 2\pi \int_0^{\pi/3} \left[\tfrac{64}{3}\sin\varphi - \tfrac{8}{3}\underbrace{\sec^3\varphi\sin\varphi}_{=\sec^2\varphi\tan\varphi}\right] d\varphi$$

$$= 2\pi\left(\int_0^{\pi/3} \tfrac{64}{3}\sin\varphi\, d\varphi - \int_0^{\pi/3} \tfrac{8}{3}\sec^2\varphi\tan\varphi\, d\varphi\right)$$

$u = \tan\varphi$
$du = \sec^2\varphi \, d\varphi$

$$= 2\pi\left(-\tfrac{64}{3}\cos\varphi\Big|_0^{\pi/3} - \tfrac{\cancel{8}^{\,4}}{3}\cdot\tfrac{1}{\cancel{2}}\tan^2\varphi\Big|_0^{\pi/3}\right)$$

$$= \tfrac{8\pi}{3}\left(-16\left(\cos\tfrac{\pi}{3} - \cos 0\right) - \left(\left(\tan\tfrac{\pi}{3}\right)^2 - (\tan 0)^2\right)\right)$$

$$= \tfrac{8\pi}{3}\left(-16\left(\tfrac{1}{2} - 1\right) - (3 - 0)\right)$$

$$= \tfrac{8\pi}{3}(5) = \tfrac{40\pi}{3}$$

Above $z = 2\sqrt{3}$: ρ is between $\rho = 2\sqrt{3}\sec\varphi$ and $\rho = 4$

$$V_{2\sqrt{3}} = \int_0^{2\pi}\int_0^{\pi/6}\int_{2\sqrt{3}\sec\varphi}^{4} \rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta$$

$$= \frac{2\pi}{3}\int_0^{\pi/6}\left(64\sin\varphi - 24\sqrt{3}\sec^3\varphi\sin\varphi\right)d\varphi$$

$$= \frac{8\pi}{3}\left(-16\cos\varphi\Big|_0^{\pi/6} - 3\sqrt{3}\tan^2\varphi\Big|_0^{\pi/6}\right)$$

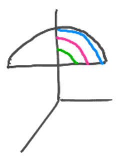
$$= \frac{8\pi}{3}\left(16 - 9\sqrt{3}\right) = \frac{128\pi}{3} - 24\sqrt{3}\,\pi$$

Volume between: $V_2 - V_{2\sqrt{3}} = \frac{40\pi}{3} - \frac{128\pi}{3} + 24\sqrt{3}\pi = \boxed{\pi\left(24\sqrt{3} - \frac{88}{3}\right)}$

Note: We usually integrate $d\rho\, d\varphi\, d\theta$. Let's switch to $d\varphi\, d\rho\, d\theta$ for V_2 above.

We want constants to bound ρ: $2 \le \rho \le 4$

We need to write φ in terms of ρ.



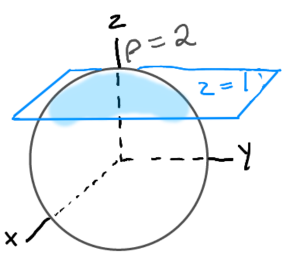
For fixed ρ, find limits on φ.

$\rho = k_1 \quad 0 \le \varphi \le$ intersection with $z=2$
$\rho = k_2 \quad 0 \le \varphi \le$ intersection with $z=2$
$\rho = k_3 \quad 0 \le \varphi \le$ intersection with $z=2$

Solve $\rho\cos\varphi = 2$ for φ in terms of ρ: $\varphi = \cos^{-1}\left(\frac{2}{\rho}\right)$

This gives $\displaystyle\int_0^{2\pi}\int_2^4\int_0^{\cos^{-1}(\frac{2}{\rho})} \rho^2\sin\varphi\, d\varphi\, d\rho\, d\theta.$

Ex. 9 Write triple integrals for the volume above the plane $z=1$ and below the sphere $\rho=2$ in rectangular, cylindrical, and spherical coordinates.



Spherical: Find φ when $z=1$ and $\rho=2$ intersect.

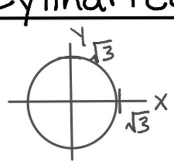
$$2\cos\varphi = 1$$
$$\varphi = \frac{\pi}{3}$$

ρ in terms of φ for the plane $z=1$.

$$\rho\cos\varphi = 1$$
$$\rho = \sec\varphi$$

$$\boxed{V = \int_0^{2\pi}\int_0^{\pi/3}\int_{\sec\varphi}^{2} \rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta}$$

Cylindrical: Intersection of $\rho=2$ with $z=1$:



$$\rho^2 = x^2 + y^2 + z^2$$
$$4 = x^2 + y^2 + 1$$
$$3 = x^2 + y^2$$
$$\Rightarrow 0 \le r \le \sqrt{3} \quad \text{and} \quad 0 \le \theta \le 2\pi$$

Need to find the bounds on z:

Plane: $z=1$

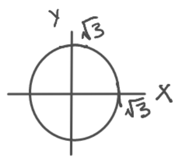
Sphere: $\underbrace{x^2+y^2}_{r^2}+z^2=4$

$$r^2+z^2=4$$

$$z=\sqrt{4-r^2}$$

$$\boxed{V=\int_0^{2\pi}\int_0^{\sqrt{3}}\int_1^{\sqrt{4-r^2}} r\,dz\,dr\,d\theta}$$

Rectangular: Need bounds on x, y, and z.



(From above)

$\underline{z}$: On plane, $z=1$

On sphere, $x^2+y^2+z^2=4$

$$z=\sqrt{4-x^2-y^2}$$

$\underline{y}$: $x^2+y^2=3 \Rightarrow -\sqrt{3-x^2}\le y\le\sqrt{3-x^2}$

$\underline{x}$: $-\sqrt{3}\le x\le\sqrt{3}$

$$\boxed{V=\int_{-\sqrt{3}}^{\sqrt{3}}\int_{-\sqrt{3-x^2}}^{\sqrt{3-x^2}}\int_0^{\sqrt{4-x^2-y^2}} dz\,dy\,dx}$$

By symmetry: $V=4\int_0^{\sqrt{3}}\int_0^{\sqrt{3-x^2}}\int_0^{\sqrt{4-x^2-y^2}} dz\,dy\,dx$