§16.6 Integrals for Mass Calculations

Formulas: $\rho(x,y)$ is density over an area/region.
$\rho(x,y,z)$ is density over a volume/solid.
Can use a constant ρ not dependent on x,y,z.

Mass, $m = \iint_R \rho(x,y)\,dA$ or $m = \iiint_D \rho(x,y,z)\,dV$

$$\bar{x} = \frac{M_y}{m} = \frac{1}{m}\iint_R x\rho(x,y)\,dA \qquad \bar{x} = \frac{M_{yz}}{m} = \frac{1}{m}\iiint_D x\rho(x,y,z)\,dV$$

$$\bar{y} = \frac{M_x}{m} = \frac{1}{m}\iint_R y\rho(x,y)\,dA \qquad \bar{y} = \frac{M_{xz}}{m} = \frac{1}{m}\iiint_D y\rho(x,y,z)\,dV$$

$$\bar{z} = \frac{M_{xy}}{m} = \frac{1}{m}\iiint_D z\rho(x,y,z)\,dV$$

M_x and M_y are the moments about the x- and y-axes respectively.

M_{xy}, M_{xz}, and M_{yz} are the moments about the xy-, xz-, and yz-planes respectively.

Center of mass/Centroid: $(\bar{x}, \bar{y}, \bar{z})$

Note: When integrating in polar or cylindrical coordinates, remember

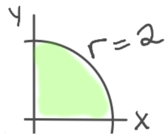
$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

In spherical coordinates,

$$x = \rho\sin\varphi\cos\theta$$
$$y = \rho\sin\varphi\sin\theta$$
$$z = \rho\cos\varphi$$

Ex.1 Find the centroid of the constant density region $R = \{(r,\theta) : 0 \le r \le 2,\ 0 \le \theta \le \frac{\pi}{2}\}$.

Let k be the constant density. Use polar coordinates!



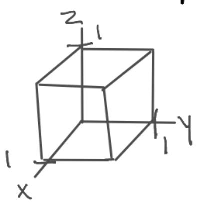
$$m = \int_0^{\pi/2}\int_0^2 k\,r\,dr\,d\theta = \frac{\pi}{2}\cdot k\cdot \frac{1}{2}r^2\Big|_0^2 = \pi k$$

$$\bar{x} = \frac{1}{m}\iint_R xk\,dA$$
$$= \frac{1}{\pi \cancel{k}}\int_0^{\pi/2}\int_0^2 (r\cos\theta)\cdot\cancel{k}\cdot r\,dr\,d\theta$$
$$= \frac{1}{\pi}\left[\int_0^{\pi/2}\cos\theta\,d\theta\right]\left[\int_0^2 r^2\,dr\right]$$
$$= \frac{1}{\pi}\left[\sin\theta\right]_0^{\pi/2}\left[\frac{1}{3}r^3\right]_0^2 = \frac{8}{3\pi}$$

$$\bar{y} = \frac{1}{m}\iint_R yk\,dA$$
$$= \frac{1}{\pi \cancel{k}}\int_0^{\pi/2}\int_0^2 (r\sin\theta)\cdot\cancel{k}\cdot r\,dr\,d\theta$$
$$= \frac{1}{\pi}\left[-\cos\theta\right]_0^{\pi/2}\left[\frac{1}{3}r^3\right]_0^2$$
$$= \frac{8}{3\pi}$$

$$\boxed{\left(\frac{8}{3\pi}, \frac{8}{3\pi}\right)}$$

Ex. 2 Find the coordinates of the center of mass of the interior of the cube in the first octant formed by the planes $x=1$, $y=1$ and $z=1$ with $\rho(x,y,z) = 2+x+y+z$.



$$m = \int_0^1\int_0^1\int_0^1 (2+x+y+z)\,dz\,dx\,dy \quad \text{(or any order)}$$
$$= \int_0^1\int_0^1 \left[2z + xz + yz + \tfrac{1}{2}z^2\right]_{z=0}^{z=1} dx\,dy$$
$$= \int_0^1\int_0^1 \left(\tfrac{5}{2} + x + y\right) dx\,dy$$
$$= \int_0^1 \left[\tfrac{5}{2}x + \tfrac{1}{2}x^2 + yx\right]_{x=0}^{x=1} dy$$
$$= \int_0^1 (3+y)\,dy$$
$$= 3y + \tfrac{1}{2}y^2 \Big|_0^1$$
$$= \frac{7}{2}$$

$$\bar{x} = \frac{1}{m}\int_0^1\int_0^1\int_0^1 x(2+x+y+z)\,dx\,dy\,dz$$
$$= \frac{2}{7}\int_0^1\int_0^1 \left[x^2 + \tfrac{1}{3}x^3 + \tfrac{1}{2}x^2y + \tfrac{1}{2}x^2z\right]_{x=0}^{x=1} dy\,dz$$
$$= \frac{2}{7}\int_0^1\int_0^1 \left(\tfrac{4}{3} + \tfrac{1}{2}y + \tfrac{1}{2}z\right) dy\,dz$$
$$= \frac{2}{7}\int_0^1 \left[\tfrac{4}{3}y + \tfrac{1}{4}y^2 + \tfrac{1}{2}yz\right]_{y=0}^{y=1} dz$$
$$= \frac{2}{7}\int_0^1 \left(\tfrac{19}{12} + \tfrac{1}{2}z\right) dz$$
$$= \frac{2}{7}\left[\tfrac{19}{12}z + \tfrac{1}{4}z^2\right]_0^1$$
$$= \frac{\cancel{2}^{\,1}}{7}\cdot\frac{\cancel{22}^{\,11}}{\cancel{12}_{\,\cancel{6}\,3}} = \frac{11}{21}$$

Note: Since the coefficients on $x, y,$ and z in $\rho(x,y,z)$ are the same and the limits on $x, y,$ and z are the same, we'll get $\bar{x} = \bar{y} = \bar{z}$

$$\boxed{\left(\frac{11}{21}, \frac{11}{21}, \frac{11}{21}\right)}$$