§17.1 Vector Fields

Recall:

(Pre-MA261) function	Vector valued function	Function of several variables
Ex. $y = f(x)$	Ex. $\vec{r}(t) = \langle f(t), g(t) \rangle$	Ex. $z = f(x,y)$
Input: Scalar	Input: Scalar	Input: Point
Output: Scalar	Output: Vector	Output: Scalar

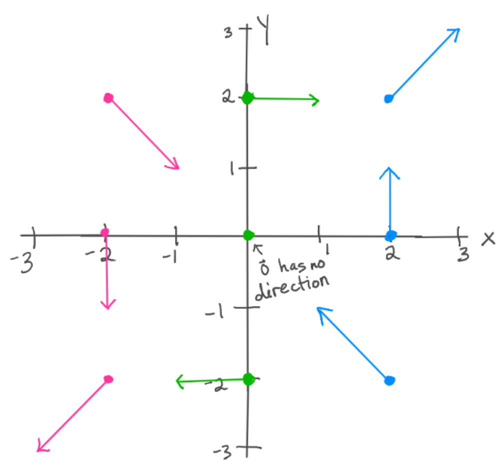
Def. Vector fields assign a vector to each point.

Input: Point
Output: Vector

We already encountered a vector field. The gradient, ∇f, of a function f gives the vector in the direction of maximum increase at any point.

Ex.1 Sketch vectors at 9 points of $\vec{F}(x,y) = \langle \frac{y}{2}, \frac{x}{2} \rangle$.

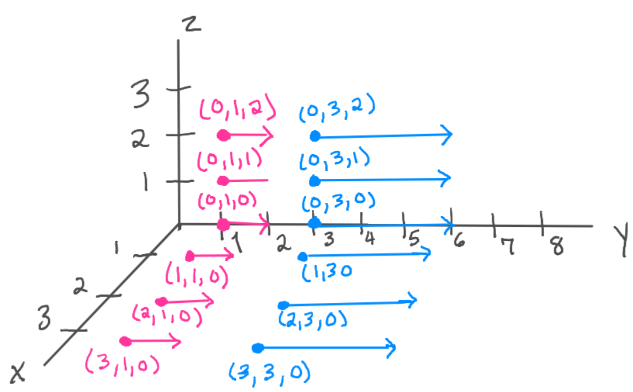
(x,y)	$\vec{F}(x,y)$
$(-2,-2)$	$\langle -1,-1 \rangle$
$(-2,0)$	$\langle 0,-1 \rangle$
$(-2,2)$	$\langle 1,-1 \rangle$
$(0,-2)$	$\langle -1,0 \rangle$
$(0,0)$	$\langle 0,0 \rangle$
$(0,2)$	$\langle 1,0 \rangle$
$(2,-2)$	$\langle -1,1 \rangle$
$(2,0)$	$\langle 0,1 \rangle$
$(2,2)$	$\langle 1,1 \rangle$



Note: We can find a vector at any point. These are a few examples.

Ex.2 Sketch the vector field $\vec{F}(x,y,z) = y\,\vec{j}$.

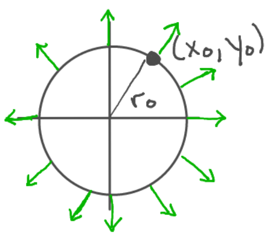
$= \langle 0, y, 0 \rangle$



Def Let $\vec{r} = \langle x, y \rangle$. Then $\vec{F} = f(x,y)\,\vec{r}$ is a radial vector field.

Specifically, if $\vec{F}(x,y) = \frac{\vec{r}}{|\vec{r}|^p} = \frac{\langle x, y \rangle}{|\vec{r}|^p}$, p is any number, then at every point except for the origin, the vectors are directed outward from the origin.

Consider a point (x_0, y_0). Then we can draw a circle of radius $\sqrt{x_0^2 + y_0^2} = |\langle x_0, y_0 \rangle| = |\vec{r}_0|$. At any point on this circle, we see that $|\vec{F}| = \frac{1}{|\vec{r}_0|^{p-1}}$ which means that the magnitude of the vector $\vec{F}(x,y)$ at every point on this circle is the same.



This is why we call the vector field "radial."

The definition is similar for a vector field $\vec{F}(x,y,z) = f(x,y,z)\,\vec{r}$.

Def. If $\vec{F} = \nabla\varphi$, we say $\vec{F}$ is a gradient field and φ is a potential function for $\vec{F}$.

Ex.3 Find the gradient field $\vec{F} = \nabla\varphi$ where $\varphi(x,y,z) = e^{x^2+y}$

$$\vec{F}(x,y,z) = \nabla\varphi(x,y,z) = \langle \varphi_x, \varphi_y, \varphi_z \rangle$$
$$= \boxed{\langle 2xe^{x^2+y}, e^{x^2+y}, 0 \rangle}$$

Ex.4 Find a potential function for $\vec{F}(x,y) = \langle 2x + y, x + 1 \rangle$.

Let's start with $\varphi_y = x + 1$. These are constants with respect to y, so two terms of φ should be $xy + y$.
However, $\frac{\partial}{\partial x}[xy + y] = y \neq 2x + y$.

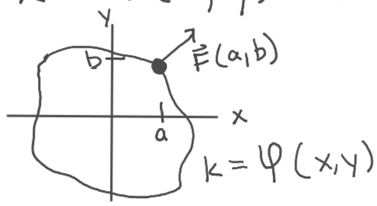
We need to add something to $xy + y$ whose derivative is $2x$ and is constant with respect to y.
x^2 satisfies this.

Then a potential function for $\vec{F}$ is $\varphi(x,y) = x^2 + xy + y$.

Check: $\nabla\varphi = \langle 2x + y, x + 1 \rangle = \vec{F}$ ✓

Def. If $\vec{F} = \nabla\varphi$, then the level curves of φ are called equipotential curves (surfaces).

Note: If $\vec{F}(x,y) = \nabla\varphi(x,y)$ and $\varphi(a,b) = k$, then the vector $\vec{F}(a,b)$ is orthogonal to the level curve $k = \varphi(x,y)$ at the point (a,b).



Ex.5 Consider $\vec{F} = \langle \frac{y}{2}, -\frac{x}{2} \rangle$ and $C = \{(x,y): y - x^2 = 1\}$.

(a) Determine any points along the curve C at which $\vec{F}$ is tangent to C.

Think of C has a level curve of $g(x,y) = y - x^2$. Then if $\vec{F}$ is tangent to C at (a,b), $\nabla g(a,b) \bullet \vec{F}(a,b) = 0$. This means we need to solve

$$\nabla g(x,y) \bullet \vec{F}(x,y) = 0$$

$$\langle -2x, 1 \rangle \bullet \langle \tfrac{y}{2}, -\tfrac{x}{2} \rangle = 0$$

$$-xy - \frac{x}{2} = 0$$

$$\underbrace{-x}_{x=0}\underbrace{\left(y + \tfrac{1}{2}\right)}_{y=-1/2} = 0$$

$x = 0$ or $y = -1/2$

Now, use $y - x^2 = 1$ to find the points.

$\underline{x=0}$: $y - 0 = 1$, $y = 1$ $\quad\boxed{(0,1)}$

$\underline{y = -\frac{1}{2}}$: $-\frac{1}{2} - x^2 = 1$, $-x^2 = \frac{3}{2}$

Not possible.

(b) Determine any points along the curve C at which $\vec{F}$ is normal to C.

$\vec{F}$ is normal to C at (a,b) if $\vec{F}(a,b)$ is parallel to $\nabla g(a,b)$, so we need to solve $\vec{F}(x,y) = k\,\nabla g(x,y)$ where k is a constant.

$$\left\langle \frac{y}{2}, -\frac{x}{2} \right\rangle = k\langle -2x, 1\rangle$$

$\Rightarrow$ $\frac{y}{2} = -2kx$ $\underline{\underline{\text{and}}}$ $-\frac{x}{2} = k$

$\frac{y}{-4x} = k$ $\underline{\text{Note}}$: $x=y=0$ is trivial.

$$\frac{y}{-4x} = -\frac{x}{2}$$

$$2y = 4x^2$$

$$y = 2x^2 \quad \text{(Similar to using Lagrange)}$$

Now, plug in to $y - x^2 = 1$

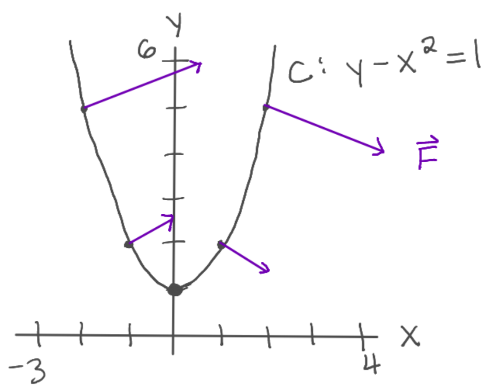
$$2x^2 - x^2 = 1$$

$$x^2 = 1$$

$$x = \pm 1$$

Plug these values of x back in to $y = 2x^2$.
Then $\vec{F}$ is normal to C at $\boxed{(1, 2) \text{ and } (-1, 2)}$

(c) Sketch C and a few representative vectors of $\vec{F}$ on C.



(x,y)	$\left\langle \frac{y}{2}, -\frac{x}{2} \right\rangle$
$(-2,5)$	$\langle \frac{5}{2}, 1\rangle$
$(-1,2)$	$\langle 1, \frac{1}{2}\rangle$
$(1,2)$	$\langle 1, -\frac{1}{2}\rangle$
$(2,5)$	$\langle \frac{5}{2}, -1\rangle$