§17.2 Line Integrals (or Integrals over a Curve)

Recall: Length of a curve C given by $\vec{r}(t)$, $a \le t \le b$, is
$L = \int_a^b |\vec{r}\,'(t)|\,dt$

Arc length function is $s(t) = \int_a^t |\vec{r}\,'(\tau)|\,d\tau$

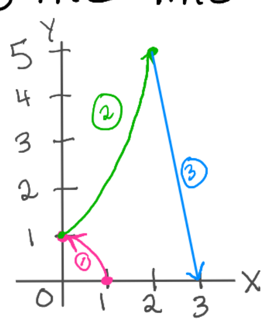
$$\Rightarrow \frac{ds}{dt} = |\vec{r}\,'(t)|$$

$$\Rightarrow ds = |\vec{r}\,'(t)|\,dt$$

Def If f is a scalar valued function and C is a curve given by $\vec{r}(t)$ for $a \le t \le b$, then the line integral of f over C is $\boxed{\int_C f\,ds = \int_a^b f(t)\,|\vec{r}\,'(t)|\,dt}$

Note: This works for any number of variables. Specifically, it works for $f(x,y)$ with $\vec{r}(t) = \langle x(t), y(t)\rangle$ and for $f(x,y,z)$ with $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$.

Ex.1 Evaluate $\int_C x\,ds$ where C consists of
① the part of the unit circle from $(1,0)$ to $(0,1)$ in Quadrant I
② the parabola $y = x^2+1$ from $(0,1)$ to $(2,5)$, and
③ the line segment from $(2,5)$ to $(3,0)$.



We need to parameterize each part of the curve.

① We can parameterize a circle of radius k by $\vec{r}(t) = \langle k\cos t, k\sin t\rangle$, $0 \le t \le 2\pi$. For this segment of the unit circle, we have $\vec{r}_1(t) = \langle \cos t, \sin t\rangle$, $0 \le t \le \frac{\pi}{2}$

② When we are given a curve in terms of one coordinate, we can make that coordinate the parameter: $\vec{r}_2(t) = \langle \underset{x(t)}{t}, \underset{y(t)}{t^2+1}\rangle$, $0 \le t \le 2$

$5 = t^2+1$
$t = 2$

③ We parameterize the line segment from $P(x_1, y_1)$ (with position vector $\langle x_1, y_1\rangle$) to $Q(x_2, y_2)$ (with $\langle x_2, y_2\rangle$) by $\vec{r}(t) = (1-t)\langle x_1, y_1\rangle + t\langle x_2, y_2\rangle$

$$= \langle x_1 + t(x_2 - x_1),\ y_1 + t(y_2 - y_1)\rangle, \quad 0 \le t \le 1$$

$$\vec{r}_3(t) = (1-t)\langle 2,5\rangle + t\langle 3,0\rangle$$
$$= \langle 2 + t(3-2),\ 5 + t(0-5)\rangle$$
$$= \langle 2+t,\ 5-5t\rangle, \quad 0 \le t \le 1$$

Now that the curves are parameterized, we have to evaluate $\int_C f\,ds = \int_{①} f\,ds + \int_{②} f\,ds + \int_{③} f\,ds$ where $f(x,y) = x$

① $\vec{r}_1(t) = \langle \cos t, \sin t\rangle = \langle x(t), y(t)\rangle, \quad 0 \le t \le \frac{\pi}{2}$

Then $f(t) = f(x(t), y(t)) = x(t) = \cos t$

$\vec{r}_1{}'(t) = \langle -\sin t, \cos t\rangle$ and $|\vec{r}_1{}'(t)| = \sqrt{\sin^2 t + \cos^2 t} = 1$

$$\int_{①} f\,ds = \int_0^{\pi/2} f(t)\,|\vec{r}_1{}'(t)|\,dt$$
$$= \int_0^{\pi/2} \cos t\,(1)\,dt$$
$$= \sin t\,\Big|_0^{\pi/2} = 1$$

Note: We can also parameterize this part of the circle using $x^2 + y^2 = 1 \Rightarrow x = \sqrt{1-y^2}$ to get

$\vec{r}(t) = \langle \sqrt{1-t^2}, t\rangle, \quad 0 \le t \le 1$

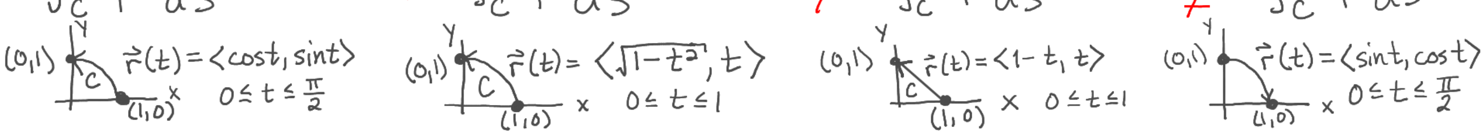
Then $f(t) = \sqrt{1-t^2}$ and $\vec{r}\,'(t) = \left\langle \frac{-t}{\sqrt{1-t^2}}, 1\right\rangle$

so $|\vec{r}\,'(t)| = \sqrt{\frac{t^2}{1-t^2} + 1} = \sqrt{\frac{\cancel{t^2} + 1 - \cancel{t^2}}{1-t^2}} = \frac{1}{\sqrt{1-t^2}}$.

Then $\int_{①} f\,ds = \int_0^1 \sqrt{1-t^2} \cdot \frac{1}{\sqrt{1-t^2}}\,dt = \int_0^1 dt = 1.$

As long as two parameterizations trace the exact same curves (not just having the same endpoints) then the line integrals are equal.

$$\int_C f\,ds \quad = \quad \int_C f\,ds \quad \ne \quad \int_C f\,ds \quad \underset{\text{and}}{\ne} \quad \int_C f\,ds$$



② $\vec{r}_2(t) = \langle t, t^2+1\rangle = \langle x(t), y(t)\rangle$, $0 \le t \le 2$

Then $f(t) = f(x(t), y(t)) = x(t) = t$

$\vec{r}_2'(t) = \langle 1, 2t\rangle$ and $|\vec{r}_2'(t)| = \sqrt{1+4t^2}$

$$\int_{②} f\, ds = \int_0^2 f(t)\,|\vec{r}_2'(t)|\, dt$$
$$= \int_0^2 t\sqrt{1+4t^2}\, dt$$
$$= \frac{1}{8}\cdot\frac{2}{3}(1+4t^2)^{3/2}\Big|_0^2$$
$$= \frac{1}{12}(17\sqrt{17}-1)$$

③ $\vec{r}_3(t) = \langle 2+t, 5-5t\rangle = \langle x(t), y(t)\rangle$, $0 \le t \le 1$

Then $f(t) = f(x(t), y(t)) = x(t) = 2+t$

$\vec{r}_3'(t) = \langle 1, -5\rangle$ and $|\vec{r}_3'(t)| = \sqrt{1+25} = \sqrt{26}$

$$\int_{③} f\, ds = \int_0^1 f(t)\,|\vec{r}_3'(t)|\, dt$$
$$= \int_0^1 (2+t)\sqrt{26}\, dt$$
$$= \sqrt{26}\left(2t+\frac{1}{2}t^2\right)\Big|_0^1$$
$$= \frac{5\sqrt{26}}{2}$$

Finally: $$\int_C f\, ds = \int_{①} f\, ds + \int_{②} f\, ds + \int_{③} f\, ds$$
$$= \boxed{1 + \frac{17\sqrt{17}-1}{12} + \frac{5\sqrt{26}}{2}}$$

Ex.2 Evaluate the line integral $\int_C x e^{yz}\, ds$ where C is given by $\vec{r}(t) = \langle t, 2t, -2t\rangle$ for $0 \le t \le 2$.

We're given the parameterization, so we need to find the different components:

$f(t) = f(t, 2t, -2t) = t e^{(2t)(-2t)} = t e^{-4t^2}$, $0 \le t \le 2$

$\vec{r}'(t) = \langle 1, 2, -2\rangle \Rightarrow |\vec{r}'(t)| = \sqrt{1+4+4} = 3$

$$\int_C f\, ds = \int_0^2 t e^{-4t^2}\cdot 3\, dt = -\frac{1}{8}e^{-4t^2}\Big|_0^2 = -\frac{1}{8}(e^{-16}-1)$$
$$= \boxed{\frac{1-e^{-16}}{8}}$$

Def If $\vec{F}$ is a vector field and C is a curve parameterized by arclength s and $\vec{T}$ is the unit tangent vector, then the line integral of $\vec{F}$ over C is

$$\boxed{\int_C \vec{F} \bullet \vec{T}\, ds = \int_a^b \vec{F} \bullet \vec{r}\,'(t)\, dt = \int_C \vec{F} \bullet d\vec{r}}$$

$$= \searrow \int_C \vec{F} \bullet \underbrace{\frac{\vec{r}\,'(t)}{\cancel{|\vec{r}\,'(t)|}}}_{=\vec{T}} \underbrace{\cancel{|\vec{r}\,'(t)|}\,dt}_{=ds} \nearrow =$$

where $\vec{r}(t)$ parameterizes C.

If C is closed and $\vec{F}$ is in $\mathbb{R}^3$, then $\boxed{\int_C \vec{F} \bullet \vec{T}\, ds}$ is called the circulation of $\vec{F}$ on C.

Ex. 3 Evaluate $\int_C \vec{F} \bullet \vec{T}\, ds$ for $\vec{F} = \langle -y, x \rangle$ on the parabola $y = x^2$ from $(0,0)$ to $(1,1)$.

$\vec{r}(t) = \langle t, t^2 \rangle, \quad 0 \le t \le 1 \quad \Rightarrow \quad \vec{F}(t) = \langle -t^2, t \rangle$

$\vec{r}\,'(t) = \langle 1, 2t \rangle$

$$\begin{aligned}
\int_C \vec{F} \bullet T\, ds &= \int_0^1 \vec{F} \bullet \vec{r}\,'(t)\, dt \\
&= \int_0^1 \langle -t^2, t \rangle \bullet \langle 1, 2t \rangle\, dt \\
&= \int_0^1 (-t^2 + 2t^2)\, dt \\
&= \frac{1}{3} t^3 \Big|_0^1 = \boxed{\frac{1}{3}}
\end{aligned}$$

Def. The work done by a force (vector) field $\vec{F}$ to move an object along a curve C in the positive direction is $\boxed{W = \int_C \vec{F} \bullet \vec{T}\, ds}$

Ex.4 Find the work required to move an object through $\vec{F} = \langle x, y, z \rangle$ on the tilted ellipse $\vec{r}(t) = \langle 4\cos t, 4\sin t, 4\cos t \rangle$ $0 \le t \le 2\pi$

$$\vec{F}(t) = \langle 4\cos t, 4\sin t, 4\cos t \rangle$$
$$\vec{r}\,'(t) = \langle -4\sin t, 4\cos t, -4\sin t \rangle$$
$$W = \int_C \vec{F} \bullet \vec{T}\, ds = \int_0^{2\pi} \vec{F} \bullet \vec{r}\,'(t)\, dt$$
$$= \int_0^{2\pi} (-16\cos t\sin t + 16\sin t\cos t - 16\cos t\sin t)\, dt$$
$$= -16 \int_0^{2\pi} \cos t \sin t\, dt$$

$u = \sin t$
$du = \cos t\, dt$

$$= -8\sin^2 t \Big|_0^{2\pi}$$
$$= \boxed{0}$$

If C is a curve parameterized by $\vec{r}(t) = \langle x(t), y(t) \rangle$ and $\vec{F}(x,y) = \langle f(x,y), g(x,y) \rangle$, then we can write $\int_C \vec{F} \bullet d\vec{r}$ as

$$\int_C \vec{F} \bullet \vec{r}\,'(t)\, dt = \int_C \langle f(x,y), g(x,y) \rangle \bullet \langle x'(t), y'(t) \rangle\, dt$$
$$= \int_C \big(f(x,y)\, x'(t) + g(x,y)\, y'(t)\big)\, dt$$
$$= \int_C f(x,y)\, \underline{x'(t)\, dt} + g(x,y)\, \underline{y'(t)\, dt}$$

Note: $x'(t) = \frac{dx}{dt} \Rightarrow x'(t)\, dt = dx$
$y'(t) = \frac{dy}{dt} \Rightarrow y'(t)\, dt = dy$

$$= \int_C f(x,y)\, dx + g(x,y)\, dy$$

Similarly, if $\vec{F} = \langle f, g, h \rangle$ and $\vec{r}(t) = \langle x, y, z \rangle$, then

$$\int_C \vec{F} \bullet d\vec{r} = \int_C f\, dx + g\, dy + h\, dz$$

Ex. 5 Evaluate $\int_C x^2 dx + dy + y\,dz$ where C is the curve $\vec{r}(t) = \langle t^2, 2t, t\rangle$ for $0 \le t \le 2$.

$$\vec{r}'(t) = \langle 2t, 2, 1\rangle$$

$$\int_C \underbrace{x^2}_{t^2} \underbrace{dx}_{x'(t)dt} + \underbrace{dy}_{y'(t)dt} + \underbrace{y}_{2t} \underbrace{dz}_{z'(t)dt}$$

$$= \int_0^2 \left((t^2)^2 x'(t) + y'(t) + 2t\, z'(t)\right) dt$$

$$= \int_0^2 (2t^5 + 2 + 2t)\, dt$$

$$= \frac{2}{6} t^6 + 2t + t^2 \Big|_0^2$$

$$= \frac{64}{3} + 4 + 4$$

$$= \frac{64 + 48}{3} = \boxed{\frac{112}{3}}$$

Note: If we tried to evaluate without rewriting dx, dy, and dz, we may see $\vec{r}(0) = \langle \underset{x_1}{0}, \underset{y_1}{0}, \underset{z_1}{0}\rangle$ and $\vec{r}(2) = \langle \underset{x_2}{4}, \underset{y_2}{4}, \underset{z_2}{2}\rangle$ to try the following:

$$\int_C x^2 dx + dy + y\,dz \overset{?}{=} \int_0^4 x^2 dx + \int_0^4 dy + \int_0^2 y\,dz$$

$$= \frac{1}{3}x^3 \Big|_{x=0}^{x=4} + y\Big|_{y=0}^{y=4} + yz\Big|_{z=0}^{z=2}$$

$$\ne \frac{64}{3} + 4 + \underbrace{2y}$$

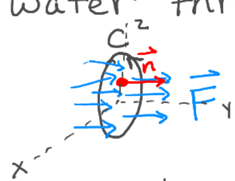
We should not have a variable after evaluating.

It is very important that we evaluate along the curve by using $\vec{r}(t)$.

Def The <u>flux</u> of a vector field $\vec{F}$ through a <u>closed</u> curve C where C does not intersect itself (<u>not</u>) is given by $\boxed{\int_C \vec{F} \bullet \vec{n}\, ds}$ where $\vec{n}$ is the unit normal to the curve.

If C is a <u>closed</u> curve with counterclockwise orientation, then $\vec{n}$ is the outward normal, and the flux gives the outward flux across C.

Note: You can visualize flux through a closed curve as water through the cross-section of a pipe.



If $\vec{F}$ is orthogonal to $\vec{n}$, then no water flows through the pipe, so the flux is 0. (Dot product!)

C in xz-plane
$\vec{n}$ in y direction

How can we compute the flux $\int_C \vec{F} \bullet \vec{n}\, ds$?

We are going to focus on flux when $\vec{F} = \langle f, g \rangle$ and C is given by $\vec{r}(t) = \langle x(t), y(t) \rangle$, $a \le t \le b$.

$\vec{n}$ is the unit normal given by $\vec{n} = \vec{T} \times \vec{k}$ where $\vec{T}$ is the unit tangent vector to C (in the xy-plane) and $\vec{k} = \langle 0, 0, 1 \rangle$.

$$\vec{T} = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \frac{\langle x'(t), y'(t), 0 \rangle}{|\vec{r}\,'(t)|} = \left\langle \frac{x'(t)}{|\vec{r}\,'(t)|}, \frac{y'(t)}{|\vec{r}\,'(t)|}, 0 \right\rangle = \langle T_x, T_y, 0 \rangle$$

$$\Rightarrow \vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ T_x & T_y & 0 \\ 0 & 0 & 1 \end{vmatrix} = \langle T_y, -T_x, 0 \rangle \sim \langle T_y, -T_x \rangle$$

$$\Rightarrow \int_C \vec{F} \bullet \vec{n}\, ds = \int_C \langle f, g \rangle \bullet \langle T_y, -T_x \rangle\, ds$$

$$= \int_C (f\,T_y - g\,T_x)\, ds$$

$$= \int_C \left(f \cdot \frac{y'(t)}{\cancel{|\vec{r}\,'(t)|}} - g \cdot \frac{x'(t)}{\cancel{|\vec{r}\,'(t)|}}\right) \cancel{|\vec{r}\,'(t)|}\, dt$$

$$\boxed{\int_C \vec{F} \bullet \vec{n}\, ds = \int_C f\, dy - g\, dx}$$