§17.3 Conservative Vector Fields

Def. A vector field $\vec{F}$ is conservative if there exists a scalar function φ such that $\vec{F} = \nabla\varphi$.

Recall: φ is called the potential function for $\vec{F}$.

In $\mathbb{R}^2$, $\vec{F}(x,y) = \langle f, g \rangle$ is conservative if $\boxed{\dfrac{\partial f}{\partial y} = \dfrac{\partial g}{\partial x}}$

In $\mathbb{R}^3$, $\vec{F}(x,y,z) = \langle f, g, h \rangle$ is conservative if

$$\boxed{\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}, \quad \frac{\partial f}{\partial z} = \frac{\partial h}{\partial x}, \quad \text{and} \quad \frac{\partial g}{\partial z} = \frac{\partial h}{\partial y}}$$

Ex.1 Is $\vec{F} = \langle x, y \rangle$ conservative?

$f(x,y) = x$ $\qquad g(x,y) = y$

$f_y(x,y) = 0$ $\qquad g_x(x,y) = 0$

$\dfrac{\partial f}{\partial y} = \dfrac{\partial g}{\partial x}$, so $\vec{F}$ is $\boxed{\text{conservative.}}$

Ex.2 Is $\vec{F} = \langle x^2 + yz, xz + 2y, z^3 + xyz \rangle$ conservative?

$f = x^2 + yz$ $\qquad g = xz + 2y$ $\qquad h = z^3 + xyz$

$f_y = z$ $\qquad g_x = z$ $\qquad h_x = yz$

$f_z = y$ $\qquad g_z = x$ $\qquad h_y = xz$

$f_y = g_x$, but $f_z \neq h_x$ and $g_z \neq h_y$, so $\boxed{\text{not conservative}}$

Ex.3 Determine a potential function φ of the conservative vector field $\vec{F} = \langle x^2 + y^2, 2xy + y^2 \rangle$.

Note: $f = x^2 + y^2$ $\qquad g = 2xy + y^2$

$f_y = 2y \quad = \quad g_x = 2y$ ✓

If φ is a potential function of $\vec{F}$, then $\vec{F} = \nabla\varphi$,

so $\langle x^2 + y^2, 2xy + y^2 \rangle = \langle \varphi_x, \varphi_y \rangle$

$\Rightarrow$ ① $\varphi_x = x^2 + y^2$ and ② $\varphi_y = 2xy + y^2$

Since φ_x is the derivative with respect to x, we will integrate ① with respect to x.

$$\varphi_x = x^2 + y^2$$
$$\varphi = \int (x^2 + y^2)\, dx$$
$$③ \quad \varphi = \tfrac{1}{3}x^3 + xy^2 + c(y)$$

Because we integrated with respect to x, the constant of integration is <u>any</u> function of y.

Now we need to find $c(y)$. We don't have information about φ itself, but we know ② $\varphi_y = 2xy + y^2$.

Take the derivative of ③ with respect to y.

$$\underbrace{\varphi_y}_{\substack{=2xy+y^2 \\ \text{by ②}}} = 2xy + c'(y)$$

We don't know anything about $c(y)$ except that it's a function of y.

$$2xy + y^2 = 2xy + c'(y)$$
$$y^2 = c'(y)$$
$$\int y^2\, dy = c(y)$$

Integrate with respect to y.

$$\tfrac{1}{3}y^3 + C = c(y)$$

C is an arbitrary constant <u>number</u>.

$$\boxed{\varphi(x,y) = \tfrac{1}{3}x^3 + xy^2 + \tfrac{1}{3}y^3 + C}$$

<u>Check</u>: $\varphi_x = x^2 + y^2 = f$ ✓
$\varphi_y = 2xy + y^2 = g$ ✓

<u>Note</u>: We could also proceed by integrating $\varphi_y = 2xy + y^2$ with respect to y. Then differentiating with respect to x and using $\varphi_x = x^2 + y^2$ to find $c(x)$.

Ex.4 Find a potential function φ of conservative vector field $\vec{F} = \langle 2xz - y^2,\ z^2 - 2xy,\ x^2 + 2yz \rangle$.

Note: $f = 2xz - y^2$ $\quad$ $g = z^2 - 2xy$ $\quad$ $h = x^2 + 2yz$

$f_y = -2y$ $\quad$ $g_x = -2y$ $\quad$ $h_x = 2x$

$f_z = 2x$ $\quad$ $g_z = 2z$ $\quad$ $h_y = 2z$

$\vec{F}$ is indeed conservative.

If φ is a potential function of $\vec{F}$, $\vec{F} = \nabla\varphi$, so we know

① $\varphi_x = 2xz - y^2$ $\quad$ ② $\varphi_y = z^2 - 2xy$ $\quad$ ③ $\varphi_z = x^2 + 2yz$

We can use any function as a starting place, but I'm going to proceed $x \to y \to z$.

Integrate ① with respect to x:

$$\varphi_x = 2xz - y^2$$

$$\varphi = \int (2xz - y^2)\, dx$$

$$④ \quad \varphi = x^2 z - xy^2 + c(y,z)$$

Since φ is a function of $x, y,$ and z, when we integrate with respect to x, any function of y and z is the constant of integration.

Now, we can use ② by differentiating ④ with respect to y or we can use ③ by differentiating ④ with respect to z.

I'll use ② $\varphi_y = z^2 - 2xy$

Starting with ④ $\quad \varphi = x^2 z - xy^2 + c(y,z)$

$$\varphi_y = -2xy + c_y(y,z)$$

$$z^2 - 2xy = -2xy + c_y(y,z)$$

$$z^2 = c_y(y,z)$$

$$\int z^2\, dy = c(y,z)$$

$$yz^2 + d(z) = c(y,z)$$

$c(y,z)$ is a function of y and z, so when we integrate with respect to y, the constant of integration is a function of z.

Now, we have $\varphi = x^2 z - xy^2 + c(y,z)$

$$⑤ \quad \varphi = x^2 z - xy^2 + yz^2 + d(z)$$

Differentiating ⑤ with respect to z gives:

$$\varphi_z = x^2 + 2yz + d'(z)$$

Finally, we can use ③ $\varphi_z = x^2 + 2yz$ to find $d(z)$.

$$x^2 + 2yz = x^2 + 2yz + d'(z)$$

$$0 = d'(z)$$

$$\int 0 \, dz = d(z)$$

Arbitrary constant number. ← $C = d(z)$

$$\boxed{\varphi(x,y,z) = x^2 z - xy^2 + yz^2 + C}$$

Check: $\varphi_x = 2xz - y^2 = f$ ✓
$\varphi_y = -2xy + z^2 = g$ ✓
$\varphi_z = x^2 + 2xy = h$ ✓

Thm Fundamental Theorem for Line Integrals

If $\vec{F}$ is a conservative vector field with potential function φ (i.e. $\vec{F} = \nabla\varphi$), then

$$\boxed{\int_C \vec{F} \bullet \vec{T} \, ds = \int_C \vec{F} \bullet d\vec{r} = \varphi(B) - \varphi(A)}$$

where C is an oriented curve from point A to point B.

Note 1: If $\vec{F}$ is not conservative, then we have to proceed as in section 17.2 by parameterizing C with $\vec{r}(t)$.

Note 2: If $\vec{F}$ is conservative, this theorem tells us that if curves C_1 and C_2 have the same endpoints A and B, then $\int_{C_1} \vec{F} \bullet \vec{T} \, ds = \int_{C_2} \vec{F} \bullet \vec{T} \, ds$.

We say the line integral is "path independent" or "independent of path".

Ex.5 Given $\vec{F} = \langle x, -y \rangle$, evaluate $\int_C \vec{F} \bullet d\vec{r}$ the following ways:

(a) along the line segment from $(1,0)$ to $(0,1)$.

(b) along the quarter circle given by $\vec{r}(t) = \langle \cos t, \sin t \rangle$ with $0 \le t \le \frac{\pi}{2}$

(c) using the Fundamental Theorem for any curve from $(1,0)$ to $(0,1)$.

(a) Line segment given by $\vec{r}(t) = (1-t)\langle 1,0 \rangle + t\langle 0,1 \rangle$, $0 \le t \le 1$

$$= \langle 1-t, t \rangle$$

$$\Rightarrow \vec{r}'(t) = \langle -1, 1 \rangle$$

$$\int_C \underbrace{\vec{F}}_{\langle x,-y \rangle} \bullet d\vec{r} = \int_0^1 \langle 1-t, -t \rangle \bullet \langle -1, 1 \rangle \, dt$$

$$= \int_0^1 -1 \, dt$$

$$= -t \Big|_0^1$$

$$= \boxed{-1}$$

(b) $\vec{r}(t) = \langle \cos t, \sin t \rangle$, $0 \le t \le \frac{\pi}{2}$

$\Rightarrow \vec{r}'(t) = \langle -\sin t, \cos t \rangle$

$$\int_C \vec{F} \bullet d\vec{r} = \int_0^{\pi/2} \langle \cos t, -\sin t \rangle \bullet \langle -\sin t, \cos t \rangle \, dt$$

$$= \int_0^{\pi/2} -2\cos t \sin t \, dt$$

$u = \cos t$
$du = -\sin t \, dt$

$$= \int_0^{\pi/2} 2u \, du$$

$$= u^2 \Big|_0^{\pi/2} = \cos^2 t \Big|_0^{\pi/2} = 0 - 1 = \boxed{-1}$$

(c) $\vec{F} = \langle x, -y \rangle$ $f_y = 0$, $g_x = 0$ so $\vec{F}$ is conservative and we need to find potential φ.

$$\nabla \varphi = \vec{F} \Rightarrow \langle \varphi_x, \varphi_y \rangle = \langle x, y \rangle$$

$\varphi_x = x$

$\varphi = \int x \, dx$

$\varphi = \frac{1}{2}x^2 + c(y)$

$\varphi_y = c'(y)$

$-y = c'(y)$

$-\int y \, dy = c(y)$

$-\frac{1}{2}y^2 + C = c(y)$

$\varphi(x,y) = \frac{1}{2}x^2 - \frac{1}{2}y^2 + C$

By the Fundamental Theorem, C from $(1,0)$ to $(0,1)$

$$\int_C \vec{F} \cdot d\vec{r} = \varphi(0,1) - \varphi(1,0)$$
$$= \left(\tfrac{1}{2}\cdot 0^2 - \tfrac{1}{2}\cdot 1^2 + C\right) - \left(\tfrac{1}{2}\cdot 1^2 - \tfrac{1}{2} 0^2 + C\right)$$
$$= \left(-\tfrac{1}{2} + C\right) - \left(\tfrac{1}{2} + C\right) = \boxed{-1}$$

Note: Since $\vec{F}$ is conservative, we can see that $\int_C \vec{F} \cdot d\vec{r}$ is path independent, i.e. the parameterization of C does not affect the value of $\int_C \vec{F} \cdot d\vec{r}$.

Def Suppose C is a curve parameterized by $r(t)$, $a \le t \le b$. C is closed if $r(a) = r(b)$, and C is simple if it never intersects itself, i.e. for all $a < t_1 < t_2 < b$, $r(t_1) \neq r(t_2)$.

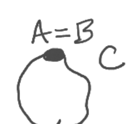
Note: no equality

simple closed | simple not closed | not simple closed | not simple not closed

Notation: If C is a simple closed curve, we write the line integral on C as $\oint_C \vec{F} \cdot d\vec{r}$

Thm Line Integrals on Closed Curves

$\vec{F}$ is a conservative vector field on a region R if and only if $\oint_C \vec{F} \cdot d\vec{r} = 0$ on all simple closed oriented curves C in R.

Why? If C is simple and closed, then 

, we have $A = B$ (starting point = ending point).
If $\vec{F}$ is conservative, $\vec{F} = \nabla \varphi$ and $\oint_C \vec{F} \cdot d\vec{r} = \varphi(B) - \varphi(A) = 0$ by the Fundamental Theorem.