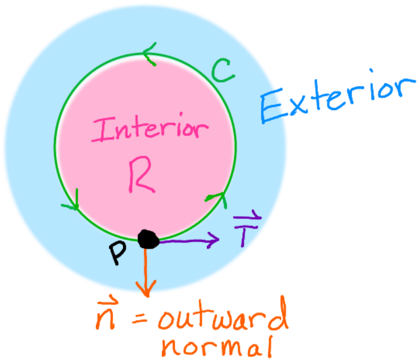
§ 17.4 Green's Theorem

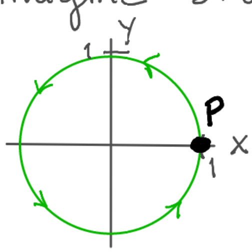
* For this section, unless otherwise stated, we will assume curves are simple, closed and oriented.
When a curve is traversed in the positive direction, the interior of the curve is to the left. At any point, there is a unique outward unit normal pointing towards the exterior of the curve.



The curve C encloses the (interior) region R. Imagine standing at point P. If you are traversing C in the positive direction, the interior should be on your left which means you are looking in the direction $\vec{T}$. Therefore, C should be traversed counterclockwise.

Recall: The circulation $\oint_C \vec{F} \cdot \vec{T}\, ds = \oint_C \vec{F} \cdot d\vec{r}$ measures the net component of $\vec{F}$ in the direction tangent to the closed curve C.

Imagine standing at point P on the circle C.



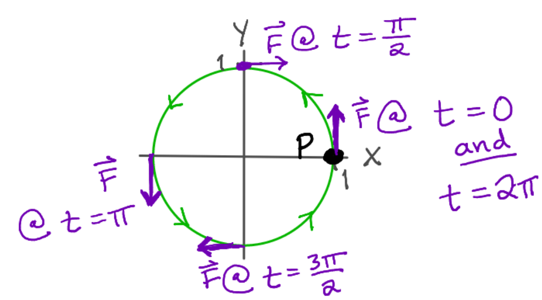
$C: \vec{r}(t) = \langle \cos t, \sin t \rangle$, $0 \le t \le 2\pi$

C will be traversed counterclockwise (positive direction here) as t increases from 0 to 2π.

Next, imagine moving along C while rotating, like the earth rotating on its axis while orbiting the sun. $\vec{F}$ will point in the direction you face as you rotate.

- $\vec{F} = \langle y, x \rangle$ is conservative.

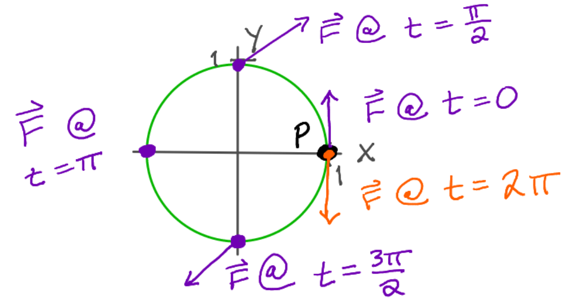
t	(x, y)	$\vec{F}$
0	(1,0)	$\langle 0,1 \rangle$
$\pi/2$	(0,1)	$\langle 1,0 \rangle$
π	(−1,0)	$\langle 0,-1 \rangle$
$3\pi/2$	(0,−1)	$\langle -1,0 \rangle$
2π	(1,0)	$\langle 0,1 \rangle$



For the conservative vector field, we start at P and end at P facing the same direction, so our net rotation is 0.
This is consistent with the previous section where we saw that the circulation on a closed curve (the net rotation) is 0.

- $\vec{F} = \langle \sin t, \pi - t \rangle$ is <u>not</u> conservative

t	(x,y)	$\vec{F}$
0	$(1,0)$	$\langle 0, \pi \rangle$
$\pi/2$	$(0,1)$	$\langle 1, \pi/2 \rangle$
π	$(-1,0)$	$\langle 0, 0 \rangle$
$3\pi/2$	$(0,-1)$	$\langle -1, -\pi/2 \rangle$
2π	$(1,0)$	$\langle 0, -\pi \rangle$



The net rotation is not 0 for this $\vec{F}$ because we start at P and end at P, <u>but</u> we end up facing the opposite direction.

$$\oint_C \vec{F} \bullet d\vec{r} = \int_0^{2\pi} \langle \sin t, \pi - t \rangle \bullet \langle -\sin t, \cos t \rangle \, dt$$
$$= \int_0^{2\pi} (-\sin^2 t + (\pi - t)\cos t) \, dt$$
$$= \int_0^{2\pi} \tfrac{1}{2}(1 - \cos 2t) \, dt + \int_0^{2\pi} (\pi - t)\cos t \, dt$$

IBP: $u = \pi - t$, $du = -dt$; $dv = \cos t\,dt$, $v = \sin t$

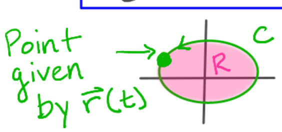
$$= \tfrac{1}{2}t - \tfrac{1}{4}\sin 2t + (\pi - t)\sin t - \cos t \Big|_0^{2\pi}$$
$$= \pi - 0 + 0 - 1 - (0 - 0 - 0 - 1) = \pi$$

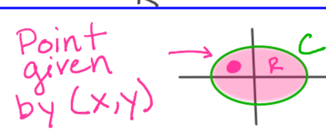
The circulation for this $\vec{F}$ on C is π (not 0).

<u>Thm</u> Green's Theorem – Circulation Form

Let C be a simple closed curve, oriented counterclockwise that encloses a region R. Then if $\vec{F} = \langle f, g \rangle$,

$$\boxed{\oint_C \vec{F} \bullet d\vec{r} = \oint_C f\,dx + g\,dy = \iint_R (g_x - f_y)\, dA}$$





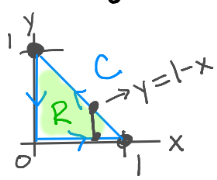
Def. If $\vec{F} = \langle f, g \rangle$, then $\boxed{\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = g_x - f_y}$ is the two-dimensional curl of $\vec{F}$.

If $g_x - f_y = 0$ in a region R, $\vec{F}$ is irrotational on R.

↳ circulation (net rotation) is 0

Ex.1 Evaluate $\int_C x^4\,dx + xy\,dy$ where C is the triangle with vertices $(0,0)$, $(1,0)$ and $(0,1)$.

(a) Using Green's Theorem:



Assume counterclockwise orientation.

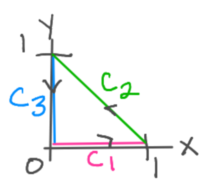
$\vec{F} = \langle x^4, xy \rangle = \langle f, g \rangle$

$\Rightarrow g_x - f_y = y - 0 = y$

By Green's Theorem,

$$\int_C x^4\,dx + xy\,dy = \iint_R (g_x - f_y)\,dA$$
$$= \int_0^1 \int_0^{1-x} y\,dy\,dx$$
$$= \int_0^1 \left[\tfrac{1}{2}y^2\right]_{0=y}^{1-x=y} dx$$
$$= \int_0^1 \tfrac{1}{2}(1-x)^2\,dx$$
$$= -\tfrac{1}{6}(1-x)^3\Big|_0^1 = \boxed{\tfrac{1}{6}}$$

(b) Not using Green's Theorem:



Parameterize each part of the curve

C_1: $\vec{r}_1(t) = \langle t, 0 \rangle$, $0 \le t \le 1$

C_2: $\vec{r}_2(t) = \langle 1-t, t \rangle$, $0 \le t \le 1$

C_3: $\vec{r}_3(t) = \langle 0, 1-t \rangle$, $0 \le t \le 1$

$$\int_C = \int_{C_1} + \int_{C_2} + \int_{C_3}$$

$$\int_{C_1} x^4\,dx + xy\,dy = \int_0^1 \left((t^4)(1) + (t)(0)(0)\right) dt$$
$$= \int_0^1 t^4\,dt = \frac{1}{5}$$

$$\int_{C_2} x^4\,dx + xy\,dy = \int_0^1 \left((1-t)^4(-1) + (1-t)t\,(1)\right) dt$$
$$= \int_0^1 \left(-(1-t)^4 + t - t^2\right) dt$$
$$= \frac{1}{5}(1-t)^5 + \frac{1}{2}t^2 - \frac{1}{3}t^3 \Big|_0^1$$
$$= 0 + \frac{1}{2} - \frac{1}{3} - \left(\frac{1}{5} + 0 - 0\right)$$
$$= \frac{1}{6} - \frac{1}{5} = -\frac{1}{30}$$

$$\int_{C_3} x^4\,dx + xy\,dy = \int_0^1 \left((0)^4(0) + (0)(1-t)(-1)\right) dt = 0$$

$$\int_C x^4\,dx + xy\,dy = \frac{1}{5} + \frac{-1}{30} + 0 = \boxed{\frac{1}{6}}$$ ■

Thm Calculating Area with Line Integrals

To find the area A of a region R bounded by C,

$$\boxed{A = \frac{1}{2}\oint_C x\,dy - y\,dx = \oint_C x\,dy = -\oint_C y\,dx}$$

Why? If $\vec{F} = \langle -y, x \rangle = \langle f, g \rangle$, $\frac{1}{2}\oint_C x\,dy - y\,dx \overset{\text{Green's}}{=} \iint_R (g_x - f_y)\,dA$

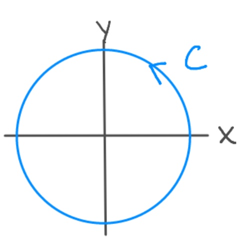
$$= \iint_R \left(\frac{\partial}{\partial x}[x] - \frac{\partial}{\partial y}[-y]\right) dA$$
$$= \iint_R 2\,dA$$
$$= 2A$$

$$\Rightarrow A = \frac{1}{2}\oint_C x\,dy - y\,dx$$

Using Green's with $\vec{F} = \langle -y, 0 \rangle$, we get $A = -\oint_C y\,dx$.

Using Green's with $\vec{F} = \langle 0, x \rangle$, we get $A = \oint_C x\,dy$.

Ex.2 Use a line integral on the boundary to find the area of a disk of radius 5.



C given by $\vec{r}(t) = \langle 5\cos t, 5\sin t \rangle$, $0 \le t \le 2\pi$

The formula from above gives

$$A = \underset{①}{\frac{1}{2}\oint_C x\,dy - y\,dx} \overset{\text{OR}}{=} \underset{②}{\oint_C x\,dy} \overset{\text{OR}}{=} \underset{③}{-\oint_C y\,dx}$$

① $A = \frac{1}{2}\int_0^{2\pi}\Big((5\cos t)(5\cos t) - (5\sin t)(-5\sin t)\Big)\,dt$

$= \frac{25}{2}\int_0^{2\pi} dt \quad = \boxed{25\pi}$

② $\int_0^{2\pi}(5\cos t)(5\cos t\,dt) = 25\int_0^{2\pi}\cos^2 t\,dt$

$= \frac{25}{2}\int_0^{2\pi}(1+\cos 2t)\,dt$

$= \frac{25}{2}\left[t + \frac{1}{2}\sin(2t)\right]_0^{2\pi} = \boxed{25\pi}$

③ similar to ② ■

Thm Green's Theorem - Flux Form

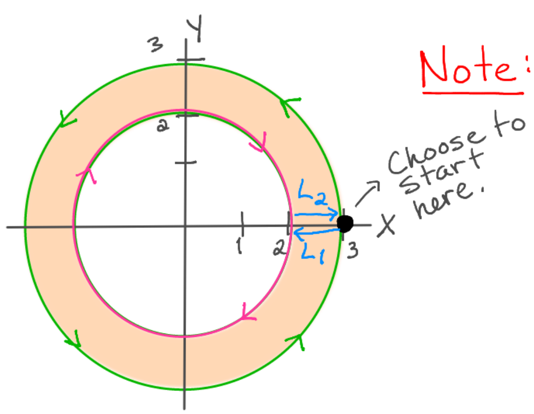
$$\boxed{\oint_C \vec{F}\bullet\vec{n}\,ds = \oint_C f\,dy - g\,dx = \iint_R (f_x + g_y)\,dA}$$

where $\vec{n}$ is the outward unit normal vector on the counterclockwise oriented curve C

Def. If $\vec{F} = \langle f, g\rangle$, then $\boxed{\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = f_x + g_y}$ is the two-dimensional divergence of $\vec{F}$.

If $f_x + g_y = 0$ in a region R, then $\vec{F}$ is source free (or divergence free) on R.

Ex.3 Use Green's theorem to find the outward flux of $\vec{F} = \langle x^3 + e^{y^2},\ y^3 - 1\rangle$ across the boundary of the annulus $R = \{(x,y) : 4 \le x^2 + y^2 \le 9\}$.



Note: The boundary of R consists of $x^2+y^2=4$ and $x^2+y^2=9$ These 2 circles are not connected to each other, so we introduce L_1 and L_2. Notice that $x^2+y^2=4$ is traversed clockwise because we want the interior R to be on the left when traversing C.

Now, we can write $\oint_C \vec{F}\bullet\vec{n}\,ds = \int_{\circlearrowleft} + \int_{L_1} + \int_{\circlearrowright} + \int_{L_2}$

L_1 and L_2 traverse the same line segment but in opposite directions, so $L_1 = -L_2$.

Recall: In calc 1, we saw $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

Similarly, we have $\int_{L_1} = -\int_{-L_1} = -\int_{L_2}$

which means
$$\int_C = \int_{\circlearrowleft} + \cancel{\left(-\int_{L_2}\right)} + \int_{\circlearrowright} + \cancel{\int_{L_2}}$$
$$= \int_{\circlearrowleft} + \int_{\circlearrowright}$$

It is important to find this proper curve C to satisfy the conditions of Green's Theorem.

Most problems in this course will have easier boundaries.

Now that we found C, we have

$$\oint_C \vec{F}\bullet\vec{n}\,ds \overset{\text{Green's}}{=} \iint_R (f_x + g_y)\,dA$$
$$= \iint_R \left(\frac{\partial}{\partial x}[x^3 + e^{y^2}] + \frac{\partial}{\partial y}[y^3 - 1]\right) dA$$
$$= \iint_R (3x^2 + 3y^2)\,dA$$

use polar

$$= \int_0^{2\pi}\int_2^3 (3r^2)\,r\,dr\,d\theta$$
$$= 2\pi\left[\frac{3}{4}r^4\right]_2^3$$
$$= \frac{3\pi}{2}(81-16) = \boxed{\frac{195\pi}{2}} \quad \blacksquare$$

Note: Without Green's Theorem:

$$\oint_C \vec{F}\bullet\vec{n}\,ds = \oint_C f\,dy - g\,dx$$

Subtract in line integral, but in the double integral, $f_x + g_y$

$$= \int_{\circlearrowleft} (x^3 + e^{y^2})\,dy - (y^3-1)\,dx$$
$$+ \int_{\circlearrowright} (x^3 + e^{y^2})\,dy - (y^3-1)\,dx$$

$x^2 + y^2 = 9 \rightarrow \vec{r}_1(t) = \langle 3\cos t,\ 3\sin t \rangle,\ 0 \le t \le 2\pi$

$x^2 + y^2 = 4 \rightarrow \vec{r}_2(t) = \langle 2\cos t,\ -2\sin t \rangle,\ 0 \le t \le 2\pi$

↑ clockwise direction

$$= \int_0^{2\pi} \left(\left((3\cos t)^3 + e^{(3\sin t)^2} \right)(3\cos t\, dt) - \left((3\sin t)^3 - 1 \right)(-3\sin t\, dt) \right)$$

$$+ \int_0^{2\pi} \left(\left((2\cos t)^3 + e^{(2\sin t)^2} \right)(2\cos t\, dt) - \left((2\sin t)^3 - 1 \right)(-2\sin t\, dt) \right)$$

Gross! $e^{4\sin^2 t}$ will be difficult to integrate since $u = 2\sin t$, $du = 2\cos t\, dt$ gives $\int e^{u^2}\, du$ which we can't integrate with the techniques we've learned.